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On the Representation of the Symmetric and Alternating Groups by Fractional Linear Substitutions

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Contents

In the present work, I deal with the task of determining all finite groups of fractional linear substitutions that are isomorphic[2] to the symmetric or alternating group of n numbers in the first degree. This task is carried out insofar as an exact outline of the desired collineation groups is gained. In the following, I call the symmetric group of n numbers S_n, the alternating group A_n.

It is sufficient to know the irreducible collineation groups; moreover, one has to consider two equivalent[3] groups, i.e., two groups which can be transformed into each other, as not being distinct.

Among the groups of fractional linear substitutions that are isomorphic [homomorphic; *Translator*] to S_n or A_n, those play a special role which can be written as groups of n! and n!/2 complete homogeneous linear substitutions.

[1] The present text is a translation of Schur, I. (1911). Über die Darstellung der symmetrischen und der alternierenden Gruppen durch gebrochene lineare substitutionen, *Journal für die reine und angewandte Mathematik*, **139**, 155–250. Translated by Marc-Felix Otto. Published with the permission of *Journal für die reine und angewandte Mathematik.*

[2] In some places, but not all, "isomorphic" must be read as "homomorphic." *Translator.*

[3] That is, isomorphic. *Translator.*

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All these groups have already been determined by Mr. Frobenius[4] by calculating the characters of the groups S_n and A_n.[5] I will show a simple method for the construction of these groups.[6]

Hence, we only have to deal with those groups in which the use of fractional linear substitutions is essential. I designate such a group as $S_n^{(g)}$ or $A_n^{(g)}$, depending on whether it is isomorphic [homomorphic; *Translator*] to S_n or A_n; correspondingly, I designate groups isomorphic [homomorphic; *Translator*] to S_n and A_n in which the fractional linear substitutions can be replaced by homogeneous linear substitutions as $S_n^{(h)}$ and $A_n^{(h)}$.

If $n < 4$, there exist no groups $S_n^{(g)}$ and $A_n^{(g)}$ at all. But if $n \geq 4$, the number of distinct (nonequivalent) irreducible groups $S_n^{(g)}$ equals the number v_n of decompositions

$$n = \nu_1 + \nu_2 + \ldots + \nu_m \qquad (\nu_1 > \nu_2 > \ldots > \nu_m > 0) \tag{1}$$

of n into different integer summands, namely a decomposition (1) corresponds to an irreducible group $S_n^{(g)}$ of the order

$$f_{\nu_1,\nu_2,\ldots,\nu_m} = 2^{[n-m/2]} \frac{n!}{\nu_1|\nu_2| \cdots \nu_3|} \prod_{\alpha>\beta} \frac{\nu_\alpha - \nu_\beta}{\nu_\alpha + \nu_\beta}$$

as I will show in the following.

Here I designate as the order of a group of fractional linear substitutions the number of variables reduced by 1, i.e., the number of variables in the corresponding homogeneous linear substitutions. For the decomposition $n = n$, one has $f_n = 2^{[n-1/2]}$. If $n = 6$, the two groups of order $f_6 = 4$ and $f_{3,2,1} = 4$ are to be considered not distinct from each other.

Mr. A. Wiman has already indicated the very interesting group of order $2^{[n-1/2]}$ in his important work, Ueber die Darstellung der symmetrischen und alternirenden Vertauschungsgruppen als Collineationsgruppen von moeglichst geringer Dimensionszahl,[7] though without specifying how this group can be composed for an arbitrary n. In Part VI, I specify a relatively easy method for the construction of this goup.

Regarding the alternating group, one has to consider the following: The group A_n has an External automorphism $A = \binom{P}{P'}$, where P' follows from P by a permutation of certain numbers in the cycles of the permutation P, e.g., of

[4] Ueber die Charaktere der symmetrischen Gruppe, *Sitzungsber. K. Preuss. Akad. Berlin* (1900), p. 516; Ueber die Charaktere der alternierenden Gruppe, *ibid.* (1903), p. 328. I have obtained the characters of the symmetric group in another way in my dissertation, Ueber eine Klasse von Matrizen, die sich einer gegebenen Matrix zuordnen lassen (Berlin, 1901).
[5] Paragraph 42 of this work shows an abstract of Frobenius' results.
[6] Ueber die Darstellung der symmetrischen Gruppe durch lineare homogene Substitutionen, *Sitzungsber. K. Preuss. Akad. Berlin* (1908), p. 664.
[7] *Math. Annalen* **52**, 243.

the numbers 1 and 2. Hence one gains from every isomorphic [homomorphic; *Translator*] collineation group K a second group K' of the same kind by substituting for the collineation of K which belongs to P the one that belongs to P' for any P. In the following, I call K and K' adjunct groups.

If one considers two adjunct groups as not different even if they are not equivalent to each other, then the number of different irreducible groups $A_n{}^{(g)}$ for $n = 4$ becomes 1 and for $n > 4$, as for the symmetric group, v_n. The irreducible group $A_n{}^{(g)}$ corresponding to the decomposition (1) equals $f_{\nu_1,\nu_2,\ldots\nu_n}$ if $n - m$ is odd and $\frac{1}{2} f_{\nu_1,\nu_2,\ldots\nu_m}$ if $n - m$ is even. However, those general rules undergo an exception in the two cases where $n = 6$ and $n = 7$. For $n = 6$, among the $v_6 = 4$ mentioned groups $A_6{}^{(g)}$, whose orders equal 4, 4, 8, 20, one has to consider the two groups of order 4, as in the group s_6, to be identical; though apart from the remaining three groups, there are six other essentially different[8] irreducible groups $A_6{}^{(g)}$ of the orders 3, 6, 6, 9, 12, 15. For $n = 7$, there are added to the $v_7 = 5$ groups $A_7{}^{(g)}$ corresponding to the general case 11 other irreducible groups of the orders 6, 6, 15, 15, 21, 21, 24, 24, 24, 24, 36.

Every group $S_n{}^{(g)}$ and $A_n{}^{(g)}$ can be written as a group of $2n!$ and $2n!/2$ homogeneous linear substitutions, respectively. This rule only fails with the alternating groups A_6 and A_7; here the minimal number of homogeneous linear substitutions by which a group $A_n{}^{(g)}$ ($n = 6, 7$) can be written can also be $3n!/2$ or $6n!/2$. This explains the exceptional status of the groups $A_6{}^{(g)}$ and $A_7{}^{(g)}$.

Of special interest is the existence of two essentially different groups $A_7{}^{(g)}$ of the order 6 to which is added a group $A_7{}^{(g)}$ of the same order. The two groups $A_7{}^{(g)}$ can be distinguished in the first place in that the one can be written as a group of 3(7!/2) homogeneous linear substitutions, the other as a group of 6(7!/2). Both these groups have been overlooked by Mr. Wiman,[9] in the examination of the collineation groups of order 6 isomorphic [homomorphic; *Translator*] with A_7.

Until now, of the groups $S_n{}^{(g)}$ and $A_n{}^{(g)}$ named above, only the binary, ternary, and quaternary groups have been known, except the group $S_n{}^{(g)}$ of the order $2^{[n-1/2]}$ and the corresponding group $A_n{}^{(g)}$ of the order $2^{[n-2/2]}$ mentioned in the work by Mr. Wiman. The binary groups $A_4{}^{(g)}$, $S_4{}^{(g)}$, and $A_5{}^{(g)}$ are first found in a geometrical outfit in the work by Mr. H. A. Schwarz, Ueber diejeniger Faelle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Funktion ihres vierten Elementes darstellt.[10] Independently, Mr. F. Klein formed these three groups in his work, Ueber binaere Fomen

[8] Two groups with conjugated complex coefficients are considered not distinct. *J. Mathematik* **139**, 2.

[9] *Ibid.*, pp. 259 ff.

[10] *J. Reine Angew. Math.* **75**, 292.

mit linearen Transformationen in sich selbst,[11] and also proved that these are the only finite binary substitution groups, disregarding two trivial cases. The existence of a ternary group $A_6^{(g)}$ was first shown by Mr. Wiman[12] by proving that a ternary collineation group already mentioned by Mr. Valentiner[13] is isomorphic [homomorphic *Translator*] to the group A_6. Among the (irreducible) quaternary collineation groups, there is one of each group $S_4^{(g)}$, $S_6^{(g)}$, $A_6^{(g)}$, $A_7^{(g)}$ and two of the groups $S_5^{(g)}$ and $A_5^{(g)}$. The groups $S_6^{(g)}$ and $A_7^{(g)}$ were first discovered by Mr. F. Klein[14] by considerations of linear geometry; each of these groups contains the group $A_6^{(g)}$ and one of the groups $S_5^{(g)}$ and $A_5^{(g)}$ as subgroups. The enumeration of all the ternary and quaternary collineation groups which are ismorphic to a symmetric or alternating group has been done by Mr. H. Maschke.[15]

In the following, I use the methods that I explained in my work, Ueber die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen.[16] To get an exact survey of all the groups $S_n^{(g)}$ and $A_n^{(g)}$, one only has to establish the representation groups of S_n and A_n and calculate the Frobenius characters of these groups.

If $n > 4$, the group S_n possesses two representation groups T_n and T'_n of the same order $2n!$ that are only isomorphic [homomorphic; *Translator*] to each other for $n = 6$. Each of these groups has an invariant subgroup M of the order 2 which is contained in the commutator of the groups, and the groups T'_n/M and T_n/M are singly[17] isomorphic to the group S_n; T_n and T'_n differ from each other in that the transpositions of S_n in T_n correspond to elements of the order 4, while those in T'_n correspond to elements of the order 2. Both groups can easily be derived from each other; I will only deal with the group T_n.

The representation group of A_n is clearly distinguished. If $n \geq 4$, but not 6 or 7, then this is a group B_n of the order $2(n!/2)$ which is contained as a subgroup in each of the groups T_n and T'_n. In contrast, the representation groups of A_6 and A_7 are of the order $6(6!/2)$ and $6(7!/2)$.

The determination of the representation groups of S_n and A_n is relatively easy if one uses a theorem on the definition of S_n and A_n as abstract finite groups by Mr. E. H. Moore, which plays an important role in the mentioned

[11] *Math. Annalen* **9**, 183.
[12] *Math. Annalen* **47**, 531.
[13] *Vidensk. Sels. Skrifter, 6. Raekke* (Copenhagen, 1889), p. 64.
[14] *Math. Annalen* **28**, 499.
[15] *Math. Annalen* **51**, 251.
[16] *J. Reine Angew. Math.* **127**, 20. Also compare to my work, Untersuchungen ueber die Darstellung der endlichen Gruppen durch gebrochene linear Substitutionen, *Ibid.*, **132**, 85. In the following, I cite the first work by D., the second one by U.
[17] In the original, em einstufig. *Translator.*

works by Mr. Wiman and H. Maschke, too.[18] The calculation of the characters of these representation groups is much harder; this required an intense study of the group T_n which on the one hand is closely related to the symmetric group, but on the other hand has a much more complicated structure. Finally, I can solve this problem by introducing a class of symmetric functions that are interesting themselves (Chapter IX).

1. THE REPRESENTATION GROUPS OF THE GROUPS S_n AND A_n

Paragraph 1. To facilitate the understanding of the following, I start with some remarks on the notions which I use.[19]

Let H be a finite group of the order h. If one assigns to the elements A, B, . . . of H the h linear substitutions (collineations) of nonvanishing determinants

$$x_\mu = \frac{a_{\mu,1}y_1 + \cdots + a_{\mu,m-1}y_{m-1} + a_{\mu,m}}{a_{m,1}y_1 + \cdots + a_{m,m-1}y_{m-1} + a_{m,m}}$$

$$x_\mu = \frac{b_{\mu,1}y_1 + \cdots + b_{\mu,m-1}y_{m-1} + b_{\mu,m}}{b_{m,1}y_1 + \cdots + b_{m,m-1}y_{m-1} + b_{m,m}}$$

then these substitutions form a representation (of the order m) of H if the product AB equals the substitution AB, which corresponds to the product AB of the elements A and B, with each two elements A, B of the group. Here, the h substitutions A, B, . . . do not need to differ from each other. If one denotes the coefficient matrices $(a_{\lambda\mu})$, $(b_{\lambda\mu})$, . . . with (A), (B), . . . , then the equation

$$(A)(B) = r_{A,B}(AB) \tag{2}$$

holds with each two elements A, B of the group, where $r_{A,B}$ is a certain constant. In the reverse case, a representation of H of fractional linear substitutions corresponds to each system of h matrices (A), (B),. . . , whose determinants are not zero and which have the property that with each two elements A, B of H there exists an equation of the form (2).

Each matrix (A), (B), . . . which corresponds to the substitutions A, B, . . . is only determined up to a constant. If these factors can be chosen such that the numbers $r_{A,B}$ all become equal to one, then the matrices (A), (B), . . .

[18] [3]Mr. de Seguier, *Co. R. Acad. Scie. Paris* (1910), **150**, 599 has determined the representation groups of S_n and A_n in another way. However, in the alternating group, Mr. de Seguier missed the important exception $n = 7$.
[19] Compare D., Introduction.

themselves form a representation of the group H which can also be interpreted as a representation of H by the even homogeneous linear substitutions

$$\begin{aligned} &(\mathrm{A}) \quad x_\mu = a_{\mu 1} y_1 + a_{\mu 2} y_2 + \cdots + a_{\mu m} y_m \\ &(\mathrm{B}) \quad x_\mu = b_{\mu 1} y_1 + b_{\mu 2} y_2 + \cdots + b_{\mu m} y_m \end{aligned} \qquad (\mu = 1,2, \ldots, m)$$

Two representations of a group by whole or fractional linear substitutions are equivalent if one representation can be transformed into the other by a whole or fractional linear transformation of the variables of a nonvanishing determinant. Moreover, a representation of mth order by whole or fractional linear substitutions is called irreducible if for none of its equivalent representations there can be found a number $k < m$ such that among the coefficients $a_{\lambda\mu}, b_{\lambda\mu}, \ldots$ of its substitutions, those become equal to zero at which $\lambda \leq k$ and $\mu > k$ or $\lambda > k$ and $\mu \leq k$.

A finite group K which contains a subgroup M consisting of invariant elements of K such that the group K/M is isomorphic to the group H in the first degree will be denoted as a group of H completed by the group M. If $K = MA' + MB' + \ldots$, the element A of H shall correspond to the complex MA', the element B to the complex B', etc. Furthermore, one has an arbitrary representation Δ' of the group K by homogeneous linear substitutions (matrices) such that to each element of the subgroup M there corresponds a matrix which only differs by a constant factor from the identity matrix.[20] If in this representation the matrices (A), (B), . . . are assigned to the elements A', B', . . . , then there exist equations of the form (2) for these matrices. Hence to each such representation Δ' of K by homogeneous linear substitutions there belongs a representation Δ of the group H by fractional linear substitutions.

The group K can always be chosen such that by this each representation of H can be established by fractional linear substitutions. A group K which has this property will be called a sufficiently completed group of H. If the order of such a group becomes the smallest possible, then I denote it as a representation group of H. Hence, if one knows a representation group K of H, one can get all the irreducible representations of K by fractional linear substitutions by determining all the irreducible representations of K by homogenous linear substitutions.

A sufficiently completed group K of H is a representation group exactly if the commutator of K contains all the elements of the subgroup M. Moreover, the commutator of each representation group, being an abstract group, is readily determined by the group H. The same is valid for the subgroup M, which I denote as the *multiplicator* of the group H. A group H whose multiplicator is of order one will be called a *closed* group.

[20] This condition is automatically satisfied with an irreducible representation.

Paragraph 2. The symmetric group S_n can be generated by the $n - 1$ transpositions

$$S_1 = (1,2), \qquad S_2 = (2,3), \ldots, \qquad S_{n-1} = (n - 1, n)$$

These transpositions satisfy the equations

$$S_\alpha^2 = E, \qquad (S_\beta S_{\beta-1}^3) = E, \qquad S_\gamma S_\delta = S_\delta S_\gamma \tag{I}$$

and we have the following theorem as shown by Mr. E. H. Moore[21]:

If one considers equations (I) as a system of defining relations between the $n - 1$ generating elements $S_1, S_2, \ldots S_{n-1}$, then the abstract group defined thereby is finite and isomorphic to the group S_n in the first degree.

Let us now consider any representation of the group S_n by collineations. A collineation with the coefficient matrix A_α will correspond to the transposition S_α; then A_α is only determined up to a constant factor. From the relations (I) there follow equations for A_α of the form

$$A_\alpha^2 = a_\alpha E \tag{3}$$

$$(A_\beta A_{\beta-1}^3) = b_\beta E \tag{4}$$

$$A_\gamma A_\delta = c_{\gamma\delta} A_\delta A_\gamma \tag{5}$$

where E is the identity matrix and a_α, b_β, and $c_{\gamma\delta}$ are certain nonzero constants. The numbers $c_{\gamma\delta}$ only appear for $n > 3$ and stay unchanged if the matrices A_α are multiplied with arbitrary constants and are therefore determined by the considered collineations alone.

It follows from (5) that

$$A_\gamma A_\delta A_{\gamma-1} = c_{\gamma\delta} A_\delta$$

Squaring on both sides yields, with (3),

$$c_{\gamma\delta}^2 = 1 \tag{6}$$

Now, in $S_\gamma = (\gamma, \gamma + 1)$, $S_\delta = (\delta, \delta + 1)$ the figures $\gamma, \gamma + 1, \delta, \delta + 1$ differ because $d' \geq \gamma' + 2$. For two more indices γ' and δ' and $\delta' \geq \gamma' + 2$, one can specify a permutation in S_n which transports the indices $\gamma, \gamma + 1, \delta, \delta + 1$ to the indices $\gamma', \gamma' + 1, \delta', \delta' + 1$. Then,

$$S^{-1} S_\gamma S = S_{\gamma'}, \qquad S^{-1} S_\delta S = S_{\delta'}$$

Correspondingly, if there is assigned a collineation with a coefficient matrix A to the permutation S, in our representation,

[21] *Proc. Lond. Math. Soc.* (1897), **28**, 357.

$$A^{-1}A_\gamma A = cA_{\gamma'}, \qquad A^{-1}A_\delta A = dA_{\delta'}$$

where c and d are certain nonzero constants. Equation (5) now yields

$$A^{-1}A_\gamma AA^{-1}A_\delta A = c_{\gamma\delta}A^{-1}A_\delta AA^{-1}A_\gamma A$$

and

$$cd \cdot A_{\gamma'}A_{\delta'} = cdc_{\gamma\delta}A_{\delta'}A_{\gamma'} = cdc_{\gamma'\delta'}A_{\delta'}A_{\gamma'}$$

Hence, $c_{\gamma\delta} = c_{\gamma'\delta'}$, i.e., all the numbers $c_{\gamma\delta}$ are the same. If we put

$$c_{\gamma\delta} = j$$

then, with (6),

$$j = \pm 1 \tag{7}$$

Moreover, from equations (4)

$$A_\beta A_{\beta+1}A_\beta = bA_{\beta+1}^{-1}A_{\beta-1}A_{\beta+1}^{-1}$$

Squaring yields readily

$$b_\beta^2 = a_\beta^2 a_{\beta+1}^2 \tag{8}$$

As we may now multiply the matrices A_α with arbitrary constants, we can fix the numbers a_α arbitrarily. First put

$$a_1 = a_2 = \ldots = a_{n-1} = j$$

Then, from (7) and (8), $b_\beta = \pm 1$, and if the matrices $B_1, B_2, \ldots B_{n-1}$ are defined by the equations

$$B_1 = A_1, \qquad B_2 = jb_1A_2, \qquad B_3 = b_1b_2A_3, \qquad B_4 = jb_1b_2b_3A_4, \ldots$$

they satisfy the relations

$$B_\alpha^2 = jE, \qquad (B_\beta B_{\beta+1})^3 = jE, \qquad B_\gamma B_\delta = jB_\delta B_\gamma$$

On the other hand, if one puts

$$a_1 = a_2 = \ldots = a_{n+1} = 1$$

and

$$C_1 = A_1, \qquad C_2 = b_1 A_1, \qquad C_3 = b_1 b_2 A_3, \qquad C_4 = b_1 b_2 b_3 A_4, \ldots$$

then it follows that

$$C_\alpha^2 = E, \qquad (C_\beta C_{\beta+1})^3 = jE, \qquad C_\gamma C_\delta = jC_\delta C_\gamma$$

Now, if $j = 1$, the relations (I) are satisfied if one substitutes $B_\alpha = C_\alpha$ for S_α. Moore's theorem yields that for $j = 1$, the fractional linear substitutions can be replaced by homogenous linear substitutions in our representation. However, this is certainly not the case if $j = -1$. For $n < 4$, the latter option is not to be considered at all.

Paragraph 3. Now it is easy to determine the representation groups on S_n. We denote with T_n the finite abstract group which is determined by the system of the defining relations

$$J^2 = E, \qquad T_\alpha^2 = J, \qquad (T_\beta T_{\beta+1})^3 = J, \qquad T_\gamma T_\delta = JT_\delta T_\gamma \qquad \text{(II)}$$

of the generating elements J, T_1, T_2, $\ldots T_{n-1}$. In the same way, $T_{n'}$ is the group defined by the relations

$$\begin{gathered} J^2 = E, \qquad T'^2_\alpha = J, \qquad (T'_\beta T'_{\beta+1})^3 = J, \\ T'_\gamma T'_\delta = JT'_\delta T'_\gamma, \qquad JT'_\alpha = T'_\alpha J \end{gathered} \qquad \text{(II}'\text{)}$$

of the generating elements $J, T'_1, T'_2, \ldots, T'_n$. J is contained as invariant element in both groups T_n and T'_n, and if one introduces the group

$$M = E + J$$

the groups T_n/M and T'_n/M become isomorphic [homomorphic; *Translator*] in the first degree to the group S_n, which can be obtained by comparing formulas (II$^-$) and (II$'$) with (I). The groups T_n and T'_n thus appear as two groups of S_n completed by the group M. Next, the equations (II) are satisfied if one substitutes for the element J the matrix jE and for the elements T_α the matrices B_α; also the equations (II$'$) are satisfied if one substitutes for the elements J and T'_α the matrices jE and C_α. Hence, each representation of the group S_n by fractional linear substitutions yields as well a representation of the group T_n as a representation of the group T'_n by GANZE linear substitutions. It follows that T_n and T'_n are to be denoted as sufficiently completed groups of S_n. As the element J is contained in the commutator on T_n and T'_n, for $n \geq 4$,

$$J = T_1 T_3 T_1^{-1} T_3^{-1}, \qquad J = T_1' T_3' T_1'^{-1} T_3'^{-1}$$

the groups T_n and T_n' are representation groups for $n \geq 4$; the multiplicator of the group S_n is of the order 2 if $n \geq 4$.[22]

One has to consider that the commutator of S_n is the alternating group A_n. As the index of this subgroup equals 2, i.e. (if $n \geq 4$), equals the order of the multiplicator of S_n, it follows that the group S_n can have maximally two representation groups not isomorphic [homomorphic; *Translator*] to each other. However, if one uses this procedure in the general case of a finite group, e.g., to get a second representation group of S_n from T_n, one is automatically led to the group T_n'. Then, if $n = 6$, S_n is a complete group,[23] hence the groups T_n and T_n' are not isomorphic [homomorphic, *Translator*] to each other for $n = 6$.[24] These two groups differ from each other in that the elements of T_n corresponding to the tranpositions of S_n are of the order 4, while those of T_n' are of the order 2. This also implies that T_6 and T_6' are isomorphic [homomorphic, *Translator*] groups. This is because the group S_n has an outer automorphism which assigns to each transposition a permutation of the form $(ab)(gd)(eh)$. In T_6, the elements corresponding to these permutations are of the order 2, which can be seen from the elements $T_1T_3T_5$ and $JT_1T_3T_5$ belonging to the permutation (12) (34) (56).[25]

We can formulate the following theorem:

I. The groups S_2 and S_3 are compact groups. However, if $n > 3$, the group S_n' possesses two representation groups T_n and T_n', each of the order $2(n!)$, which can be defined as abstract groups by the relations (II) and (II′) and T_n and T_n' are isomorphic groups only if $n = 6$.

Paragraph 4. Now I consider the alternating group A_n. This group is generated by the $n - 2$ permutations

$$A_1 = S_2 S_1 = (123), \qquad A_2 = S_3 S_1 = (12)(34), \ldots,$$
$$A_{n-2} = S_{n-1} S_1 = (12)(n-1, n)$$

which satisfy the equations

[22] It should also be proved that $j = E$ cannot follow from the relation (II) or (II′). This follows from the fact that these relations can be satisfied by matrices such that E and J are replaced by two different matrices, as we will see in Chapter IV.
[23] Compare to O. Hoelder, Bildung zusammengesetzter Gruppen, *Math. Ann.* **46**, 321.
[24] Compare to U., p. 122.
[25] It can be seen directly that T_6 and T_6' are isomorphic by showing that the elements $T_1 = T_1'T_3'T_5'$, $T_2 = T_3'T_2'T_1'T_4'T_3'T_2'T_5'T_4'T_3'$, $T_3 = T_1'T_4'T_3'T_5'T_4'$, $T_4 = T_1'T_2'T_1'T_3'T_2'T_1'T_5'$, $T_5 = T_1'T_3'T_4'T_3'T_5'T_4'T_3'$ of T'_6 satisfy the relations defining T_6.

(III)

Again, the group A_n is clearly defined as an abstract group by these relations.[26] Let there be given an arbitrary representation of A_n by collineations. If a collineation with the coefficient matrix P_n corresponds to a permutation A_n, then P_n is only determined up to a constant factor and with (III) there exist equations of the form

$$P_1^3 = a_1 E, \qquad (P_1 P_2)^3 = b_1 E, \qquad (P_1 P_\lambda) = c_\lambda E \tag{9}$$

$$P_\alpha^2 = a_\alpha E, \qquad (P_\beta P_{\beta+1})^3 = b_\beta E, \qquad P_\lambda P_\delta = c_{\lambda\delta} P_\delta P_\lambda \tag{10}$$

Equations (10) are completely analogous to equations (3)–(5) of Paragraph 2. We conclude like above, that

$$c_{\gamma\delta} = c_{24} = \pm 1, \qquad b_\beta^2 = a_\beta^3 a_{\beta+1}^3 \tag{11}$$

Moreover, from (9),

$$(P_2 P_1{}^2)^3 = (a_2 a_1 P_2{}^{-1} P_1{}^{-1})^3 = a_2{}^3 a_1{}^3 b_1{}^{-1} E$$

and

$$P_1 P_2 P_1 = b_1 P_2{}^{-1} P_1{}^{-1} P_2{}^{-1} = b_1 a_2{}^{-1} P_2{}^{-1} P_1{}^{-1} P_2$$

The last equation yields, raised to the third power,

$$P_1 P_2 P_1^2 P_2 P_1^2 P_2 P_1 = P_1 (P_2 P_1^2)^3 P_1^{-1} = b_1^3 a_2{}^{-3} (P_2{}^{-1} P_1^{-1} P_2)^3$$

Hence,

$$a_2^3 a_1^3 b_1^{-1} = b_1^3 a_2{}^{-3} a_1^{-1}, \qquad \text{i.e.,} \quad b_1^4 = a_1^4 a_2^6$$

Putting

$$\frac{b_1^2}{a_1^2 a_2^3} \tag{12}$$

then $j = \pm 1$. From $(P_1 P_\lambda)^2 = c_\lambda E$, one also gets

$$P_\lambda P_1 P_\lambda = a_\lambda P_\lambda^{-1} P_1 P_\lambda = c_\lambda P_1^{-1}$$

and, raising to the third power,

[26] E. H. Moore, *op cit.* The group A_n can be defined more elegantly by the relations

$$C_\alpha^3 = E, \qquad (C_\alpha C_\beta)^2 = E \qquad (\alpha, \beta = 1, 2, \ldots n-2, \beta > \alpha)$$

which can be show using Moore's theorem. But this definition of A_n is not so useful in this case.

$$a_\lambda^3\, a_1 = c_\lambda^3\, a_1^{-1}, \qquad \text{i.e.,} \quad c_\lambda^3 = a_1^2 a_\lambda^3 \tag{13}$$

For $n \geq 6$ and

$$k \frac{a_3 c_4}{c_3 a_4}$$

(13) yields

$$k_3 = j = \pm 1$$

Moreover, from the equations

$$(P_1P_4)^2 = c_4E, \qquad P_2P_4 = c_2^4 P_4 P_2$$

one readily obtains the equation

$$P_4P_1P_2 = c_4 c_2^4 P_1^{-1} P_2 P_4^{-1}$$

or

$$P_4P_1P_2P_4^{-1} = c_4 a_2 a_4^{-1} c_2^4 P_1^{-1} P_2^{-1}$$

This implies, by raising both sides to the third power,

$$b_1 = c_4^3 a_2^3 a_4^{-3} c_{24}^3 b_1^{-1}$$

Considering equations (11)–(13) one concludes that $c_{24} = j$; therefore, generally,

$$c_{\lambda\delta} = j$$

For $n \geq 7$, also consider the equations

$$(P_1P_\mu)^2 = c_\mu, \qquad P_3P_\mu = jP_\mu P_3 \qquad (\mu \geq 5)$$

These yield

$$P_3P_1P_\mu = c_3 P_1^{-1} P_3^{-1} P_\mu = jc_3 P_1^{-1} P_\mu P_3^{-1}$$

i.e.,

$$P_3P_1P_\mu P_3^{-1} = jc_3 a_\mu a_3^{-1} P_1^{-1} P_\mu^{-1}$$

Raising both sides to the second power, one obtains

$$c_\mu = c_3^2 a_\mu^2 a_3^{-2} c_\mu^{-1}$$

i.e.,

$$c_\mu^2 a_\mu^{-2} = c_3^2 a_3^{-2}$$

On the other hand, it follows from (13) that

$$c_\mu^3 a_\mu^{-3} = c_3^3 a_3^{-3}$$

and hence

$$\frac{c_3}{a_3} = \frac{c_5}{a_5} = \frac{c_6}{a_6} = \ldots = \frac{c_{n-2}}{a_{n-2}}$$

Also, if $n > 7$,

$$\frac{c_4}{a_4} = \frac{c_6}{a_6} = \frac{c_7}{a_7} = \ldots = \frac{c_{n-2}}{a_{n-2}}$$

Therefore, if $n > 7$,

$$\frac{c_3}{a_3} = \frac{c_4}{a_4} = \frac{c_5}{a_5} = \ldots = \frac{c_{n-2}}{a_{n-2}}$$

In particular, if $n > 7$,

$$k = j = \pm 1$$

One sees easily that the hereby introduced quantities j and k, which are connected by the equation $k_3 = j$, stay unchanged if the matrices P_1, P_2, . . . P_{n-2} are multiplied with arbitrary constants; they are determined only by the collineations to be considered. The quantity k appears only for $n > 5$ and equals j for $n > 7$. Hence, only for $n + 6$ and $n + 7$ has k an essential meaning. Later, we will see that there are representations of the groups A_6 and A_7 at which k becomes a primitive sixth root of the identity.

In order to get easier formulas, we put for $n = 4$

$$Q_1 = \sqrt[3]{\frac{i}{a_i}}\, P_1, \qquad Q_2 = j\,\frac{a_1 a_2}{b_1}\, P_2$$

Then,

$$Q_1^3 = jE, \qquad (Q_1 Q_2)^3 = jE \tag{14}$$

For $n > 4$, we put

$$Q_1 = j\,\frac{c}{a_1 a_3}\, P_1; \qquad Q_2 = j\,\frac{a_1 a_2}{b_1}\, P_2; \qquad Q_3 = \frac{1}{a_1}\,\frac{b_1}{B_2}\, a_3 P_3, \ \ldots$$

and, generally,

$$Q_{2\nu} = j\,\frac{a_1}{b_1}\,\frac{b_2 b_4 \cdots b_{2\nu-2}}{b_3 b_5 \cdots b_{2\nu-1}}\, a_{2\nu} P_{2\nu}, \qquad Q_{2\nu+1} = \frac{1}{a_1}\,\frac{b_1 b_3 \cdots b_{2\nu-1}}{b_2 b_4 \cdots b_{2\nu}}\, a_{2\nu+1} P_{2\nu+1}$$

A simple calculation yields for $n = 5$

$$Q_1^3 = Q_2^2 = Q_3^2 = (Q_1Q_2)^3 = (Q_1Q_3)^2 = (Q_2Q_3)^3 = jE \qquad (15)$$

for $n = 6$

$$Q_1^3 = Q_2^2 = Q_3^2 = Q_4^2 = (Q_1Q_2)^3 = (Q_1Q_3)^2 \qquad (16)$$
$$= (Q_2Q_3)^3 = (Q_3Q_4)^3 = jE(Q_1Q_4)^2 = kE; \qquad Q_2Q_4 = jQ_4Q_2$$

for $n = 7$

$$Q_1^3 = Q_\alpha^2 = (Q_1Q_2)^3 = (Q_1Q_3)^2$$
$$= (Q_1Q_5)^2 = (Q_\beta Q_{\beta+1})^3 = jE(Q_1Q_4)^2 = kE$$
$$Q_\gamma Q_\delta = jQ_\delta Q_\gamma$$
$$\alpha = 2, 3, 4, 5; \qquad \beta = 2, 3, 4; \qquad \gamma = 2, 3; \qquad \delta \geq \gamma + 2 \qquad (17)$$

and for $n > 7$

$$Q_1^3 = jE; \qquad (Q_1Q_2)^3 = jE; \qquad (Q_1Q_\lambda)^2 = jE$$
$$Q_\alpha^2 = jE; \qquad (Q_\beta Q_{\beta+1})^3 = jE$$
$$Q_\gamma Q_\delta = jQ_\delta Q_\gamma \qquad (18)$$

The indices α, β, γ, δ, λ in equations (18) fulfill the same conditions as in equations (III).

Paragraph 5. Now, we can easily determine the representation group of A_n.[27] Consider the representation group T_n of S_n. The (MEHRSTUFIG) isomorphism between S_n and T_n corresponds to the subgroup A_n of the order $n!/2$ of S_n, a subgroup B_n of the order $2n!/2$ of T_n. This group B_n can be generated by the elements

$$B_1 = T_2T_1, \qquad B_2 = T_3T_1, \qquad B_3 = T_4T_1, \ldots, \qquad B_{n-2} = T_{n-1}T_1$$

and from the relations (II), it immediately follows that these elements satisfy equations analogous to equations (III):

$$B_1^3 = J; \qquad (B_1B_2)^3 = J; \qquad (B_1B_\lambda)^2 = J; \qquad B_\alpha^2 = J \qquad \text{(IV)}$$
$$(B_\beta B_{\beta+1})^3 = J; \qquad B_\gamma B_\delta = JB_\delta B_\gamma$$

These equations also clearly define the group B_n as an abstract group. It readily follows from (IV) that J commutes with the elements $B_1, B_2, \ldots B_{n-2}$ and has the order 2.

[27] Theorem II of my work U. yields that the group A_n, which is a simple group if $n < 4$, only has one representation group.

The group B_n is a group of A_n completed by the group $M = E + J$ and it can be easily seen that the commutator of B_n contains the element J if $n \geq 4$.[28] If n is greater than 3, but not 6 or 7, the formulas (14), (15), and (18) indicate that equations (IV) are satisfied if one substitutes for the elements J and B_n the matrices jE and Q_n. From this it follows that, similarly as in Paragraph 3 with the group S_n, the group B_n is the representation group of A_n if $n \geq 4$ and $n = 6$ or 7.

However, equations (16) and (17) imply that the representation groups of A_6 and A_7 are certain groups of orders 6(6!/2) and 6(7!/2), also considering the equation $k_3 = j = \pm 1$. I will explore these groups more deeply in Chapter XI.

The two cases where $n = 2$ and $n = 3$, not considered so far, are of no interest for us. That is because A_2 has the order 1 and A_3 is cyclic and therefore a compact group. Defining the group B_n, we started with the group T_n. One is led to the same group if one considers the second representation group of S_n, T'_n, instead of T_n. This can be seen by showing that the elements $B_1 = JT'_2T'_1$, $B_2 = JT'_3T'_1, \ldots, B_{n-2} = JT'_{n-1}T'_1$ of T'_n satisfy the relations (IV).

If $n \geq 4$, the group B_n can be characterized in another way, too. Namely, considering that the commutator of S_n is the group A_n and that the commutator of T_n (or T'_n) contains the element J, it follows, that the group B_n is nothing but the commutator of T_n (or T'_n). Hence, we can formulate the following theorem:

II. The representation group of the alternating group A_n is, if n is greater than 3 and not 6 or 7, a group with the order $2(n!/2)$ which is isomorphic in the first degree to the commutator of any representation group of the symmetric group S_n. On the other hand, the representation groups of the groups A_6 and A_7 are of orders 6(6!/2) and 6(7!/2), respectively.

In the discussion of the representations of the group S_n by collineations, it is of no interest which one of the two representation groups is chosen. If in the following the group T_n is considered primarily, this has the following reason: The elements $A_2, A_3, \ldots A_{n-2}$ of the group A_n generate a group which is isomorphic to the group S_{n-2}. Analogously, the elements $B_2, B_3, \ldots B_{n-2}$ of B_n generate a group of S_{n-2} completed by the group M. However, equations (IV) show that this group is isomorphic to the group T_{n-2} and not to the group T'_{n-2}.

[28] This follows from the equation $B_1^{-1}B_2B_1 \cdot B_2 = JB_2 \cdot B_1^{-1}B_2B_1$.

2. ON THE CLASSIFICATION OF THE ELEMENTS OF THE GROUPS T_n AND B_n INTO CLASSES OF CONJUGATED ELEMENTS

Paragraph 6. If the permutation P of the group S_n equals the product

$$S_\alpha S_\beta S_\gamma \ldots$$

of the transpositions $S_1 = (12)$, $S_2 = (23)$, . . . , $S_{n-1} = (n - 1, n)$, then the two elements

$$T_\alpha T_\beta T_\gamma \ldots$$

and

$$JT_\alpha T_\beta T_\gamma \ldots$$

in the group T_n correspond to this permutation. We designate one of these elements by P', the other JP'. For any permutation P of S_n, we have unique fixed element P' of T_n. Hence, the $n!$ elements P' of T_n generate a complete remainder system of T_n mod M and, if the equation $PQ = R$ is satisfied for three permutations P, Q, and R, $P'\ Q'$ equals either R' or JR'. For two commuting (similar) permutations A and B, $A'\ B'$ equals either $B'\ A'$ or $JB'\ A'$. Furthermore, if P and Q are two conjugated permutations, the element P' in T_n is conjugated to at least one of the elements Q' or JQ'.

I designate a permutation P as a permutation of the first or second kind depending on whether P' and JP' are conjugated elements of T_n or not. Two similar permutations belong to the same kind.

Now, let

$$P, P_1, P_2, \ldots P_{h-1}$$

be the complete permutations similar to the given permutation P. If P is of the first kind, the $2h$ elements

$$P', JP', P_1', JP_1', \ldots P_{h-1}', JP_{h-1}'$$

generate *one* class of conjugated elements of T_n. However, if P is of the second kind, these $2h$ elements are distributed in two classes, each consisting of h elements; here, one class turns into the other one by multiplying each of its elements with J. We can distinguish these two cases in the following manner, too: In the first case, there is a permutation Q which commutes with P without Q' commuting with P', and the number of elements of T_n which commute with P' equals the number $n!/h$ of permutations of S_n commuting with P. In the second case, however, for any permutation Q which commutes with P, Q' also commutes with P', and the number of elements of T_n commuting with P' is two times the number of permutations of S_n commuting with P.

Now consider two (commuting) permutations A and B of which the first leaves the numbers $m + 1, m + 2, \ldots, n$ unchanged, the second, the numbers $1, 2, \ldots, m$. Then A can be represented as the product of the transpositions

$$S_1, S_2, \ldots S_{m-1}$$

and B as the product of the transpositions

$$S_{m+1}, S_{m+2}, \ldots S_{n-1}$$

However, if λ stands for one of the indices $1, 2, \ldots, m - 1$ and μ for one of the indices $m + 1, m + 2, \ldots, n - 1$, then

$$T_\lambda T_\mu = JT_\mu T_\lambda$$

and it is easily seen that the elements A' and B' of T_n do not commute exactly if the permutations A and B are both odd. With little effort, it can be concluded generally:

III. If A and B are two permutations of S_n of which the cycles of order greater than one have no figure in common, then the elements A' and B' of T_n do not commute only if the permutations A and B are both odd; in this case, $A'\ B' = JB'\ A'$.

Paragraph 7. With this rule the following can be proved:

IV. An even permutation is of the first kind if it has cycles of an even order and of the second kind if it only has cycles of an odd order. An odd permutation is of the first kind if it has at least two cycles of the same order ≥ 1 and of the second kind if all the orders of its cycles are distinct.

To prove this theorem, we have to distinguish four cases.

(a) The permutation P is even and contains a cycle A of even order. If $P = AB$, then, as P is an even and A an odd permutation, B becomes an odd permutation. Now, A and B are two odd permutation whose cycles (of an order greater than 1) have no figure in common. Hence, with III,

$$A'B' = JB'A'$$

or

$$A'^{-1}(A'B')A' = JA'B'$$

As P' equals either $A'B'$ or $JA'B'$, it follows that $A'^{-1}P'A' = JP'$; hence P is a permutation of the first kind.

(b) The permutation P consists of cycles of odd order only. Then the order a of P is odd. Hence $P'^\alpha = J^\alpha$, where α equals zero or one, and $(JP')^\alpha = J^\alpha J^\alpha = j^{\alpha+1}$. It follows that the orders of P' and JP' are distinct and hence P' and JP' cannot be conjugated elements, i.e., P is a permutation of the second kind.

(c) P is an odd permutation which contains two cycles A and B of the same order ≥ 1. For example,

$$A = (\alpha_1, \alpha_2, \ldots \alpha_m), \qquad B = (\beta_1, \beta_2, \ldots \beta_m)$$

Putting

$$C = (\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots \alpha_m, \beta_m)$$

then $C^2 = AB$ and $P = C^2D$, where D is the product of the cycles different from A and B. As P is an odd and C^2 an even permutation, D becomes an odd permutation whose cycles have no figure in common with the cycles of the odd permutation C. Therefore we again get $D'C' = JC'D'$ and

$$C'^{-1}(C'^2D')C' = C'D'C' = JC'^2D'$$

As P' differs from $C'^2\ D'$ only by a factor J, it follows that $C'^{-1}P'C' = JP'$, i.e., P is of the first kind.

(d) The odd permutation P consists of r cycles $C_1, C_2, \ldots C_r$ whose orders $c_1, c_2, \ldots c_r$, are distinct. Then $P = C_1C_2 \ldots C_r$ commutes only with the $c_1c_2, \ldots c_r$, permutations

$$C_1^{\gamma_1}C_2^{\gamma_2} \ldots C_r^{\gamma_r} \qquad (\gamma_\rho = 0, 1, \ldots, c_\rho - 1)$$

If s denotes the number of odd numbers among $c_1, c_2, \ldots c_r$, then, as P is an odd permutation, s is odd. Considering the elements $C_1', C_2', C_3', \ldots C_r'$ of T_n, then for each two indices ρ and σ

$$C_\rho'C_\sigma' = C_\sigma'C_\rho' \qquad \text{or} \qquad C_\rho'C_\sigma = JC_\sigma'C_\rho'$$

namely they obey the following rule: If c_ρ for a fixed ρ is odd, i.e., the permutation C_ρ is even, then each ρ satisfies the first equation. However, if c_ρ is an even number and C_ρ an odd permutation, the second equation holds only for those $s - 1$ number σ which are distinct from ρ and for which the numbers c_σ are also even. As $s - 1$ is even, one immediately sees that each element C_σ' commutes with the product $C_1', C_2', \ldots C_r'$ and hence with the element P', too, which differs from this product only by a factor J. Hence, P' commutes with the $2c_1c_2 \cdots c_r$ elements

$$J^\beta C_1'^{\gamma_1}C_2'^{\gamma_2} \cdots C_r'^{\gamma} \qquad (\beta = 0, 1; \quad \gamma_\rho = 0, 1, \ldots c_\rho - 1)$$

Thus P' and JP' cannot be conjugated elements.

Paragraph 8. We can determine the number k_n' of classes of conjugate elements easily now.

I call a decomposition

$$n = \nu_1 + \nu_2 + \cdots + \nu_m \qquad (\nu_1 \geq \nu_2 \geq \cdots \geq \nu_m)$$

of the number n in even positive summands an *even* or an *odd decomposition* depending on whether the number of the odd numbers among $\nu_1, \nu_2, \ldots \nu_m$ is even or odd. Furthermore, I denote with k_n the number of all decompositions of n into equal or different summands, g_n denoting the number of even and u_n the number of odd decompositions of n into distinct summands. Moreover, I think of v_n as the number of decompositions of n into equal or distinct odd summands. As we know, the number v_n also determines the number of decompositions of n into distinct summands[29]; hence

$$v_n = g_n + u_n \tag{19}$$

Now, the number of classes of conjugated permutations of S_n equals k_n. To a class of permutations of the first kind of S_n there corresponds only one class of conjugated elements in T_n. However, to each class of permutations of the second kind of S_n there correspond two classes of conjugated elements of T_n. As the number of the last mentioned classes of S_n equals $v_n + u_n$ (using Theorem IV), the *desired number* k'_n of classes of T_n becomes

$$k_n - v_n - u_n + 2(v_n + u_n) = k_n + v_n + u_n$$

Also considering equation (19), this yields

$$k'_n = k_n + g_n + 2u_n \tag{20}$$

I also state the following. The numbers k_n and v_n can be calculated in a familiar manner using easy recursive equations.[30] If one knows v_n, however, g_n and u_n can be derived easily. Namely, putting

$$d_n = g_n - u_n, \qquad d_0 = 1$$

we find that (19) yields

$$g_n = \frac{1}{2}(v_n + d_n), \qquad u_n = \frac{1}{2}(v_n - d_n)$$

However, if $|x| < 1$,

$$\sum_0^\infty d_n x^n = (1 + x)(1 - x^2)(1 + x^3)(1 - x^4) \cdots$$

[29] Compare to Bachmann, *Analytische Zahlentheorie*, p. 30.
[30] Compare to Bachmann, *ibid.*, p. 28, 44.

i.e.,

$$\sum_{0}^{\infty} (-1)^n d_n x^n = (1 - x)(1 - x^2)(1 - x^3)(1 - x^4) \cdots$$

Using an equation stated by Euler,[31]

$$\prod_{1}^{\infty} (1 - x^{\lambda}) = \sum_{-\infty}^{+\infty} (-1)^{\nu} x^{(3\nu^2+\nu)/2}$$

this yields that $d_n = 0$ if n is not of the form $(3\nu^2 + \nu)/2$ and that $d_n = (-1)(\nu^2 + \nu)/2$ if $n = (3\nu^2 + \nu)/2$.

Here are some values of g_n and u_n:

$$\begin{array}{lllll} g_1 = 1, & g_2 = 0, & g_3 = 1, & g_4 = 1, & g_5 = 1, \\ g_6 = 2, & g_7 = 2, & g_8 = 3, & g_9 = 4, & g_{10} = 5 \\ u_1 = 0, & u_2 = 1, & u_3 = 1, & u_4 = 1, & u_5 = 2, \\ u_6 = 2, & u_7 = 3, & u_8 = 3, & u_9 = 4, & u_{10} = 5 \end{array}$$

Paragraph 9. Next I consider the subgroup B_n of T_n, which corresponds to the subgroup A_n of S_n.

One gets the group B_n by calculating the elements P' and JP' of T_n for all the $n!/2$ even permutations P. To a class C of h conjugated permutations of the group A_n corresponds either only one class of $2h$ conjugated elements in the group B_n or two classes of h elements each, where one class can be turned into the other one by multiplying each of its elements with J. If P is a permutation of the class C, then the first or the second case appears depending on whether P' and JP' are conjugated elements of B_n or not.

In order to carry out the classification of elements of B_n into classes of conjugated elements, we have to decide for which of the even permutations P the elements P' and JP' are conjugated with respect to the group B_n. Such a permutation P is characterized by the fact that one can find an *even* permutation Q commuting with P such that P' and Q' do not commute but rather satisfy the equation $P'Q' = JQ'P'$.

If P is a permutation of the second kind (i.e., a permutation of an odd order), then P' and JP' are not even conjugated within T_n, and hence not in B_n either. Therefore we only have to examine the even permutations of the first kind, i.e., those even permutations among whose cycles there appear some of an even order. I will show now:

[31] Compare to Bachmann, *ibid.*, p. 24

V. If P is a permutation of the first kind, then P' and JP' are conjugated elements of B_n always and only if P contains at least two cycles of the same order $m \geq 1$.

Namely, if A is a cycle of even order of P, then, as we have seen before, $A'^{-1}P'A' = JP'$. Moreover, P shall contain two cycles

$$B = (\beta_1, \beta_2, \ldots \beta_m), \qquad C = (\gamma_1, \gamma_2, \ldots \gamma_m)$$

of the same order $m \geq 1$; one of the cycles B and C may equal A. We put

$$D = (\beta_1, \gamma_1, \beta_2, \gamma_2, \ldots \beta_m, \gamma_m)$$

such that $D^2 = BC$. If $P = BCF = D^2F$, then F, as P is even, is an even permutation; according to Theorem III, $D'F' = F'D'$ and hence

$$D'^{-1}(D'^2F')D' = D'^2F'$$

It follows that $D'^{-1}P'D' = P'$ and therefore

$$(A'D')^{-1}P'(A'D') = JP'$$

As A and D are odd permutations, AD is contained in A_n and $A'D'$ in B_n. Hence P' and JP' are conjugated in B_n.

Let P be composed of r cycles $C_1, C_2, \ldots C_r$ with distinct orders $c_1, c_2, \ldots c_r$. Then P commutes only with the $c_1, c_2, \ldots c_r$ permutations

$$C_1^{\gamma_1}C_2^{\gamma_2} \cdots C_r^{\gamma_r} \qquad (\gamma_\rho = 0, 1, \ldots, c_\rho - 1)$$

within S_n. Among these permutations, those are even at which the sum of all γ_ρ corresponding to even c_ρ is an even number. As P is an even permutation, the number s of even numbers among the c_ρ is even. Similarly to case (d) in Paragraph 7, we conclude that the element P' commutes or does not commute with the element C'_ρ or T_n depending on whether c_ρ is odd or even. This yields that P' always commutes with the element

$$C_1'^{\gamma_1}C_2'^{\gamma_2} \cdots C_r'^{\gamma_r}$$

if the corresponding permutation $C_1^{\gamma_1}C_2^{\gamma_2} \cdots C_r^{\gamma_r}$ is even. As a result, there is no even permutation Q commuting with P such that P' and Q' become noncommuting elements. This implies that P' and JP' cannot be conjugated elements of B_n, q.e.d.

Paragraph 10. In the following it will be shown that, *if l_n denotes the number of classes of conjugated elements of the group A_n, the corresponding number for the group B_n becomes*

$$l'_n = l_n + 2g_n + u_n \tag{21}$$

where g_n and u_n have the same meaning as in Paragraph 8.

Considering a class C of h conjugated even permutations of the group S_n, one sees that they also build up a class of conjugated elements in the group A_n. There appears an exception only if the cycles of each permutation of C have distinct orders; in this case, the h permutations of C in the group A_n can be divided into two classes of $\frac{1}{2}h$ conjugated elements each. One class can be turned into the other one by transforming its elements using an arbitrary odd permutation.[32]

If v_n' is the number of decompositions of n into distinct summands, the even permutations of the second kind in the group A_n are distributed over $v_n + v_n'$ classes of conjugated elements. To these classes there correspond exactly $2(v_n + v_n')$ classes of conjugated elements in the group B_n. Denoting with g_n' the number of even decompositions of n into distinct summands (among which may also appear even numbers), one has in A_n exactly g_n' classes of conjugated permutations belonging to the *first* kind and whose cycles have distinct orders. By Theorem V, there correspond exactly $2g_n'$ classes of conjugated elements in the group B_n to these g_n' classes. In contrast, to each of the remaining $l_n - (v_n + v_n' + g_n')$ classes of A_n there belongs only one class within B_n. Hence,

$$l_n' = l_n - (v_n + v_n' + g_n') + 2(v_n + v_n' + g_n') = l_n + v_n + v_n' + g_n'$$

However, as $v_n' + g_n' = g_n$ and $v_n = g_n + u_n$ equation (21) follows immediately.

I will call those even permutations whose cycles have distinct orders *permutations of the third kind*. Such a permutation is also of the first kind if there appear even numbers in the orders of its cycles and of the second kind if all these orders are odd. There are only two permutations P and Q of the third kind at which P' and Q' are conjugated within T_n, but not within B_n. Two such elements of B_n will be called *conjugated elements*. Analogously, I call two permutations of A_n that are conjugated within S_n, but not within A_n, *conjugated permutations*.

3. ON THE ASSIGNMENT OF THE ELEMENTS OF THE GROUPS S_n AND T_n

Paragraph 11. We have not yet made a convention on which of the two elements of T_n corresponding to a permutation P of S_n shall be designated as P'. It is essential to fix the name. Hereby, we try to achieve that *for each two permutations P and Q which are conjugated within S_n or A_n, P' and Q' become conjugated elements of T_n or B_n.*

[32] Compare to Frobenius, Ueber die Charaktere der alternierenden Gruppe, *Sitzungsber. K. Preuss. Akad. Berlin* (1901), p. 303.

A cycle

$$C_{\mu,\nu} = (\mu, \mu + 1, \ldots \mu + \nu - 1)$$

of order ν can be represented as

$$C_{\mu,\nu} = S_{\mu+\nu-2} S_{\mu+\nu-3} \cdots S_{\mu}$$

using the transpositions $S_\alpha = (\alpha, \alpha + 1)$. *Then, we will define the element*

$$C'_{\mu,\nu} = T_{\mu+\nu-2} T_{\mu+\nu-3} \cdots T_{\mu}$$

If ν is odd, among the two elements $C'_{\mu,\nu}$ and $JC'_{\mu,\nu}$, only one is conjugated to the special element $C'_{1,\nu}$ within T_n, according to Theorem IV. However, it is easy to see that this happens with the element $C'_{\mu,\nu}$. Indeed, the two groups T_ν and $\overline{T}_\nu$ which are generated by the elements $T_1, T_2, \ldots T_{\nu-1}$ and $T_\mu, T_{\mu+1}, \ldots T_{\mu+\nu-2}$, respectively, are isomorphic according to the relations (II) that define the group T_n. Namely, one gets an isomorphism between these groups by mapping the generating element $T_{1+\rho}$ of T_ν to the element $T_{\mu+\rho}$ of $\overline{T}_\nu$. This implies that $C'_{1,\nu}$ and $C'_{\mu,\nu}$ have the same order ν', where ν' is equal to ν or 2ν.[33] If $C'_{1,\nu}$ and $JC'_{\mu,\nu}$ were conjugated elements of T_n, they would be of the same order, which is not the case as ν is odd.

If A is an arbitrary cycle of *odd* order ν, only one of the two elements of T_n that belong to A is conjugated to the element $C'_{1,\nu}$. *This element I designate as A'. Moreover, if*

$$P = A_1 A_2 \cdots A_m$$

is a permutation whose cycles $A_1, A_2, \ldots A_m$ have only odd orders, I put[34]

$$P' = A'_1 A'_2 \cdots A'_m \tag{22}$$

Here, the elements $A'_1, A'_2, \ldots A'_m$, according to Theorem III, commute with each other because the permutations A_μ are even. Therefore, the sequence of the factors A'_μ in (22) can be changed arbitrarily. The order of the element P' is nothing but the smallest divisor of the orders of $A'_1, A'_2, \ldots A'_m$. One can see easily that the element to be called Q' is conjugated to the elements P' of T_n or B_n if Q is a permutation conjugated to P within S_n or A'_n.

C shall be a cycle with even order which satisfies the condition that it only contains the numbers $\mu, \mu + 1, \ldots \mu + \nu - 1$ (in an arbitrary order). Then, C can be represented as a product of the transpositions $S_\mu, S_{\mu+1}, \ldots S_{\mu+\nu-2}$ like the cycle $C_{\mu,\nu}$. Therefore, the two elements of T_n belonging to

[33] It can be shown that $\nu' + \nu$ or $\nu' + 2\nu$, depending on whether $(-1)^{(\nu 2-1)/8}$ equals 1 or -1. If ν is even, the order of $C'_{\mu,\nu}$ becomes ν if $\nu = 8\lambda$ or $\nu = 8\lambda + 6$, but 2ν if $\nu = 8\lambda + 2$ or $\nu = 8\lambda + 4$.

[34] If $P = E$, I also put $P' = E$, of course.

C are contained in the already considered group $\overline{T}_\nu$. According to Theorem IV, there is only one of these two elements conjugated to the element $C'_{\mu,\nu}$ with respect to the group $\overline{T}_\nu$. *The element characterized hereby will be designated as* C'. Then with two distinct cycles B and C of the order ν only containing the numbers μ, $\mu + 1, \dots \mu + \nu - 1$, B' and C' are conjugated to each other in the group $\overline{T}_\nu$.

Next we consider the permutations P with the m cycles

$$C_1 = C_{1,\nu_1} = (1, 2, \dots \nu_1),$$

$$C_2 = C_{\nu_1+1,\nu_2} = (\nu_1 + 1, \nu_1 + 2, \dots \nu_1 + \nu_2), \dots$$

where $\nu_1 > \nu_2 > \dots > \nu_m \geq 1$ and there shall be even numbers among the ν_μ such that P is a permutation of second or third kind. Then we have

$$P = C_1 C_2 \cdots C_m$$

Correspondingly, I put

$$P' = C'_1 C'_2 \cdots C'_m$$

In this equation, the order of the factors may not be changed arbitrarily any more. It has to be mentioned, however, that those factors C'_μ with which the ν_μ are odd can be ordered freely. The element P' stays unchanged if one writes first the factors C'_μ with an odd ν_μ and then the factors with even ν_μ such that their values decrease.

For any permutation Q whose m cycles have the same orders $\nu_1, \nu_2, \cdots \nu_m$ as those of P, P and Q are similar permutations. Among the two elements of T_n that belong to Q only one is (according to the Theorems IV and V) conjugated to the element P' with respect to T_n if P and Q are odd permutations, and, if P and Q are even, only one is conjugated with P' with respect to B_n. *I designate the element that satisfies the first or the second condition as* Q'.

We have now made a particular convention for all the permutations P of second or third kind determining which element of T_n shall be called P'. We think of the designations for the permutations of the first kind as fixed. Considering that for each of these permutations P the elements P' and JP' are conjugated with respect to T_n and, if P is even, also with respect to B_n, one sees that as a matter of our conventions the condition formulated earlier is satisfied: if P and Q are two permutations being conjugated in S_n or A_n, then P' and Q' are conjugated elements of T_n or B_n.

Paragraph 12. We have to make a remark that is essential for the following. It refers to the case that the permutations can be decomposed into cycles of distinct orders.

In particular, be Q a permutation with m cycles $D_1, D_2, \ldots D_m$ of the orders $\nu_1 > \nu_2 > \cdots \nu_m$ such that the cycle D_μ only contains the numbers

$$\nu_1 + \nu_2 + \cdots \nu_{\mu-1} + 1, \ldots \nu_1 + \nu_2 + \cdots \nu_\mu \tag{23}$$

in an arbitrary order. We have already arranged which elements of T_n are to be called $Q', D'_1, D'_2, \ldots D'_m$. In any case,

$$Q' = J^\alpha D'_1 D'_2 \cdots D'_m \tag{24}$$

where α is 0 or 1. *We will examine the conditions that determine whether* $\alpha = 0$ *or* $\alpha = 1$.

I call the symmetric group consisting of all the $\nu_\mu!$ permutations of the indices (23) H_μ and the subgroup of the order $2 \cdot \nu_\mu!$ of T_n corresponding to the subgroup H_μ of S_n, K_μ. If C_μ has the same meaning as before, then C_μ and D_μ are similar permutations of H_μ; also, according to our conventions, C'_μ and D'_μ are conjugated elements of K_μ. Let H_μ be a permutation of H_μ satisfying the condition

$$H_\mu^{-1} C_\mu H_\mu = D_\mu$$

Then,

$$H_\mu'^{-1} C'_\mu H'_\mu = D'_\mu$$

If ν_μ is even, we choose H_μ to be an even permutation, which is always possible. If ν_μ is odd, however, H_μ is an even permutation if C_μ and D_μ are conjugated with the indices (23) in the alternating group, but if H_μ is odd, this is not the case. Let the number of indices μ such that H_μ is odd be equal to r and s be the number of the even numbers among the ν_μ. If ν_μ is even, H'_μ always commutes with C'_ρ and D'_ρ if $\rho = \mu$, as H_μ is even, according to Theorem III. The same is valid with an odd ν_μ associated to an even permutation H_μ. However, if ν_μ is odd and so is H_μ, then H'_μ commutes with C'_ρ and D'_ρ if $\rho = \mu$ and ν_ρ are odd; though, if ν_ρ is even,

$$H_\mu'^{-1} C'_\rho H'_\mu = JC'_\rho, \qquad H_\mu'^{-1} D'_\rho H'_\mu = JD'_\rho$$

Putting

$$H = H_1 H_2 \cdots H_m$$

it follows that

$$H' = J^\beta H'_1 H'_2 \cdots H'_m$$

where β is 0 or 1. One can see easily that, if P' denotes the product $C'_1, C'_2 \cdots C'_m$,

$$H'^{-1}P'H' = J^{rs}D_1'D_2' \cdots D_m' = J^{rs-\alpha}Q' \tag{25}$$

I claim now that *in equation (24),* α *equals 0 or 1, depending on whether r is even or odd*.

Let s be odd. Then, P and Q are odd permutations. Q' denotes the element of T_n conjugated to P'. Hence, it follows from (25) that

$$H'^{-1}P'H' = Q' = J^rD_1'D_2' \cdots D_m'$$

i.e., $\alpha \equiv r \pmod 2$. Otherwise, if s is even, P and Q are even permutations of the third kind. In this case Q' shall be conjugated to P' with respect to the group B_n. If r is even, H is an even permutation, hence, $H'^{-1}P'H' = Q'$. Equation (25) tells us that $\alpha = 0$. If r is odd, H is an even permutation and hence, $H'^{-1}P'H' = JQ'$.[35] According to (25), $\alpha = 1$.

4. GENERAL PROPERTIES OF THE CHARACTERS OF THE GROUPS T_n AND B_n

Paragraph 13. Considering an arbitrary representation of a finite group H by homogeneous linear substitutions in f variables (matrices of fth degree) and with $\chi(R)$ being the trace of the substitution corresponding to the element R of H, one designates the system of numbers $\chi(R)$ as a *character of fth degree of the group H*, according to Mr. Frobenius.[36] If the representation is irreducible, $\chi(R)$ is called a *simple* character. Two representations are equivalent exactly if they possess the same character. The number of simple characters $\chi^{(0)}(R)$, $\chi^{(1)}(R)$,. . . equals the number of classes of conjugated elements of H and these characters satisfy the relations

$$\sum \chi^{(\alpha)}(R)\chi^{(\alpha)}(R^{-1}) = h, \qquad \sum \chi^{(\alpha)}(R)\chi^{(\beta)}(R^{-1}) = 0 \tag{26}$$

where R stands for any element of H and h is the order of H.[37]

Moreover, one calls the system of numbers

$$\zeta(R) = r_0\chi^{(0)}(R) + r_1\chi^{(1)}(R) + \ldots$$

a *composed character* of H, where $r_0, r_1, \ldots$ are arbitrary integers. It follows from (26) that

[35] This follows from the fact that Q' and JQ' are conjugated within T_n, but not within T_n.
[36] This immediately implies that $X(R) = \chi(R')$, where R and R' are conjugated elements of H.
[37] Easy proofs of these theorems which have been formulated, by Mr. Frobenius in a number of works (*Sitzungsber. K. Preuss. Akad. Berlin*, 1896–1899) first can be found in two works by Mr. W. Burnside (*Acta Math.* **28**, 369, and *Proc. Lond. Math. Soc. Ser. 2* (1904), **1**, 117; also see my work, Neue Begruendung der Theorie der Gruppencharaktere, *Sitzungsber. K. Preuss. Akad. Berlin* (1905), p. 406.

$$\sum \zeta(R)\zeta(R^{-1}) = h(r_0^2 + r_1^2 + \ldots) \tag{27}$$

$\zeta(R)$ is a simple character only if this sum equals h and $\zeta(E) > 0$. If none of the numbers r_0, r_1, ... is negative, there belongs a representation of H by matrices of the order $\zeta(E)$ to $\zeta(R)$; in this case, $\zeta(R)$ is also called an *actual* character.

Next let H be one of the groups T_n or B_n and, correspondingly, let G be either S_n or A_n. If the (actual) character of fth degree $\chi(R)$ of H satisfies[38]

$$\chi(J) = j\chi(E), \qquad j = \pm 1$$

the matrices corresponding to the elements R and JR in the representation of H belonging to $\chi(R)$ differ only by a factor j, such that

$$\chi(JR) = j\chi(R) \tag{28}$$

These two matrices determine only *one* fractional linear substitution and the totality of these substitutions builds a group K isomorphic to the group G which I will call the *collineation group belonging to the character* $\chi(R)$. If k denotes the order of K, K can be written as a group of k homogeneous linear substitutions exactly if $j = 1$ or $n \leq 3$ (compare to Paragraph 2).

A character $\chi(R)$ of H satisfying equations (28) will be called a *character of the first or second kind* depending on whether $j = +1$ or $j = -1$.

If $\chi(R)$ is a simple character of the first kind of H, then the numbers

$$\overline{\chi}(P) = \chi(P') = \chi(JP')$$

build a simple character of G. In this connection, P' denotes the element of T_n or B_n associated to the permutation P of S_n or A_n. Conversely, one obtains from each character $\overline{\chi}(P)$ of G a simple character of the first kind $\chi(R)$ of H by putting the numbers $\chi(P')$ and $\chi(JP')$ equal to $\overline{\chi}(P)$. Therefore the number of simple characters of the first kind of H equals the number of simple characters of G, i.e., the number of classes of conjugated elements within the group G. With the numbers k_n, k'_n, l_n, l'_n, ν_n, g_n, and u_n having the same meaning as in Paragraphs 8 and 10, we obtain the following result:

The number of simple characters of second kind in the group T_n equals

$$k'_n - k_n = g_n + 2u_n = \nu_n + u_n$$

and in the group B_n

$$l'_n - l_n = 2g_n + u_n = \nu_n + g_n$$

As the characters of the groups S_n and A_n are already known (see

[38] This condition is automatically satisfied in the case of a simple character

Introduction), the characters of the first kind of T_n and B_n can be neglected and we only care about the characters of the second kind.

Paragraph 14. From every representation Δ of the group T_n by homogeneous linear substitutions (matrices) one can obtain a second representation Δ' by leaving the matrices of Δ corresponding to the elements of B_n unchanged and changing the sign of the remaining ones. I call Δ and Δ' *associated representations* and the corresponding characters *associated characters* of T_n. Two associated characters $\chi(T)$ and $\chi'(T)$ of T_n are marked by the fact that

$$\chi'(T) = (-1)^{\tau}\chi(T)$$

where τ is 0 or 1, depending on whether T is contained in B_n or not. Particularly, if $\chi(T) = \chi'(T)$, i.e., $\chi(T) = 0$ with all the elements T of T_n not contained in B_n, I designate $\chi(T)$ as *self-associated* or as *a two-sided character.*

For a simple, not two-sided character $\chi(T)$ of T_n, it follows from (26) that

$$\sum \chi(T)\chi(T^{-1}) = 2n!, \qquad \sum (-1)^{\tau}\chi(T)\chi(T^{-1}) = 0 \tag{29}$$

i.e.,

$$\sum \chi(B)\chi(B^{-1}) = n! \tag{30}$$

Here, T stands for any element of T_n and B for any element of B_n. For a simple two-sided character $\chi(T)$,

$$\sum \chi(B)\chi(B^{-1}) = 2n! \tag{31}$$

This implies that the numbers $\chi(B) = \phi(B)$ of the first case represent a simple character of the group B_n; in the second case,

$$\chi(B) = \psi(B) + \overline{\psi}(B)$$

where $\psi(B)$ and $\overline{\psi}(B)$ are distinct simple characters of B_n [compare to (27)]. It is easily seen that two associated characters of T_n are either both of the first or both of the second kind. Also, the characters $\phi(B)$, $\psi(B)$, and $\overline{\psi}(B)$ of B_n are of the first or second kind depending on whether the character $\chi(T)$ of T_n is a character of the first or second kind.

Among the $g_n + 2u_n$ simple characters of second kind of T_n, there shall be r two-sided ones and $2s$ not two-sided ones. As the latter appear as pairs, s is an integer. Keeping in mind that to each pair of associated characters of T_n there belongs only one simple character of B_n, but to each two-sided character of T_n, two characters of B_n, one obtains $2r + s$ simple characters of second kind of B_n in total. Using equations (26), one can see that these $2r + s$ characters are distinct; moreover, according to a theorem by Mr.

Frobenius,[39] these are all the simple characters of second kind of B_n. As the number of these characters is $2g_n + u_n$, it follows that

$$2r + s = 2g_n + u_n$$

On the other hand,

$$r + 2s = g_n + 2u_n$$

hence,

$$r = g_n, \qquad s = u_n, \qquad r + s = g_n + u_n = v_n$$

The number of two-sided (simple) characters of second kind of T_n equals the number of even decompositions of n in distinct summands.

Also, I state that the number of two-sided characters of first kind of T_n equals the number of decompositions of n in distinct odd summands.[40]

Paragraph 15. If C is an arbitrary element of T_n not contained in B_n, e.g., the element T_1 corresponding to the transposition $S_1 = (1, 2)$, one obtains an outer automorphism A of B_n by assigning to the element B of B_n the element $\overline{B} = C^{-1}BC$. Any character $\theta(B)$ of B_n thus yields a second character $\overline{\theta}(B)$ such that

$$\overline{\theta}(B) = \theta(\overline{B})$$

Two such characters are denoted as *adjunct characters*.[41] One concludes immediately that if $\theta(B)$ is a simple character of first or second kind, $\overline{\theta}(B)$ has the same property.

I will show now that *the two characters* $\psi(B)$ *and* $\overline{\psi}(B)$ of B_n developed *from a two-sided (simple) character $\chi(T)$ of T_n are adjunct.*

Consider an (irreducible) representation Δ of T_n by matrices of the degree $f = \chi(E)$ which belongs to $\chi(T)$. The matrix corresponding to the element T will be called T, too. As the representation associated to Δ is equivalent to Δ, one can name a matrix H with a nonvanishing determinant such that

$$H^{-1}TH = (-1)^{\tau}T \tag{32}$$

where τ has the same meaning as before. This yields that the matrix H^2 commutes with any matrix of Δ. As Δ is irreducible, $H^2 = aE_f$, where a is a constant and E_α denotes the identity matrix of αth degree. We can assume without reducing the validity that $a = 1$ such that $H^2 = E_f$. Hence, one can choose a matrix M with a nonvanishing determinant such that

[39] Ueber die Relationen zwischen den Charakteren einer Gruppe und denen ihrer Untergruppen, *Sitzungsber. K. Preuss. Akad. Berlin* (1898), 501.
[40] Compare to Frobenius, Ueber die Charaktere der symmetrischen Gruppe, Paragraph 6, and the dissertation of the author, Paragraph 23.
[41] The character $\overline{\theta}(B)$ does not depend on the choice of the element C.

$$M^{-1}HM = \begin{pmatrix} E_p & 0 \\ 0 & -E_p \end{pmatrix}$$

where p and q are positive integers with sum f. Substituting for the matrices T the matrices $M^{-1}TM$, one obtains a representation equivalent to Δ where the matrix $M^{-1}HM$ plays the same role as H in Δ. Hence, we can assume that

$$\begin{pmatrix} E_p & 0 \\ 0 & -E_p \end{pmatrix}$$

Equations (32) then yield that, if the elements of T_n contained in the subgroup B_n are called B, the others C, the matrices B and C in our representation Δ are of the form

$$B = \begin{pmatrix} P & 0 \\ 0 & \overline{P} \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & Q \\ \overline{Q} & 0 \end{pmatrix}$$

where P and $\overline{P}$ are quadratic matrices of the degrees p and q, Q is a matrix with p rows and q columns, and $\overline{Q}$ is a matrix with q rows and p columns. If $p = q$, the determinants of C would vanish, which is not the case. Hence, $p = q$ and $f = 2p$.

The matrices P and $\overline{P}$ obviously generate two representations of the group B_n. However, as we know that $\chi(B)$ appears as the sum of the two simple characters $\psi(B)$ and $\overline{\psi}$ (B) of B_n, these representations have to be irreducible.

We can assume that $\psi(B)$ is the trace of the matrix P and $\overline{\psi}(B)$ the trace of $\overline{P}$.

Let C be an element of T_n to which there corresponds a transposition in S_n, e.g., the transposition $S_1 = (1, 2)$. Then, the element C^2 equals J, i.e. the matrix C^2 equals jE_f, where $j = \pm 1$. Hence, we obtain $Q\overline{Q} = \overline{Q}Q = JE$. It is easy to see that the representation Δ can be replaced by an equivalent representation in which

$$C = \begin{pmatrix} 0 & E_p \\ jE_p & 0 \end{pmatrix}$$

and H stays unchanged. For C of this form, we obtain

$$C^{-1}BC = \begin{pmatrix} \overline{P} & 0 \\ 0 & P \end{pmatrix}$$

This implies, which is to be shown, that

$$\psi(C^{-1}BC) = \overline{\psi}(B), \qquad \overline{\psi}(C^{-1}BC) = \psi(B) \tag{33}$$

Paragraph 16. I also put

$$\delta(B) = \psi(B) - \overline{\psi}(B)$$

and designate the system of $n!$ numbers $\delta(B)$ as the *complement of the two-sided character* $\chi(T)$. As ψ and $\overline{\psi}$ commute, the complement $\delta(B)$ is determined only up to a sign by the character $\chi(T)$. This sign has no meaning in this context, as it suffices to know, apart from the numbers $\chi(J)$, either the numbers $\psi(B)$ or the numbers $-\delta(B)$ in order to be able to name the two characters $\psi(B)$ and $\overline{\psi}(B)$ of B_n.

The number $\delta(B)$ is nothing but the trace of the matrix HB. Hence, if one knows a representation Δ belonging to the character $\chi(T)$ and with H being a matrix satisfying the equation $H^2 = E_f$ and also the conditions (32), one only has to name the traces of the matrices HB in order to determine the complement of the character $\chi(T)$.

If $\chi(J) = jf$, then the numbers $\delta(B)$ satisfy the equations

$$\delta(JB) = j\delta(B) \tag{34}$$

and also, following from (27),

$$\sum_B \delta(B)\delta(B^{-1}) = 2n! \tag{35}$$

For *any* element C of T_n which is not contained in B_n, it follows from (33) that

$$\delta(C^{-1}BC) = -\delta(B) \tag{36}$$

In particular, if $\overline{B} = C^{-1}BC$ and B are conjugated within B_n, then

$$\delta(\overline{B}) = \delta(B) = 0 \tag{37}$$

Only if B and $\overline{B}$ can be called adjunct elements of B_n in the sense of Paragraph 10 can $\delta(B)$ not be zero. *Therefore, the complement* $\delta(B)$ *has to be determined only with such elements* B *of* B_n *to which correspond permutations of the third kind in* A_n.

Generally, let $\xi(T)$ be an arbitrary composed character of T_n being self-associated, i.e., which satisfies $\xi(T) = 0$ if T is not contained in B_n. Then there is an infinite number of different ways to put

$$\xi(T) = r_\alpha\chi^{(\alpha)}(T) + r_\beta\chi^{(\beta)} + \ldots + r_r\chi^{(\nu)}(T)$$

where

$$\chi^{(\alpha)}(T), \chi^{(\beta)}(T), \ldots \tag{38}$$

are simple, not necessarily distinct characters of T_n and r_α, r_β, $\ldots$ r_ν are integers. However, if $\xi(T)$ satisfies the condition

$$\xi(J) = j\xi(E), \qquad j = \pm 1$$

then also

$$\chi^{(\alpha)}(J) = j\chi^{(\alpha)}(E), \qquad \chi^{(\beta)}(J) = j\chi^{(\beta)}(E), \ldots \qquad \chi^{(\nu)}(J) = j\chi^{(\nu)}(E)$$

If

$$\chi^{(\alpha)}(T), \chi^{(\beta)}(T), \ldots \chi^{(\kappa)}(T)$$

are all the two-sided ones among the characters (38) and if one knows the complements

$$\delta^{(\alpha)}(B), \delta^{(\beta)}(B), \ldots \delta^{(\kappa)}(B)$$

then I designate *as a complement of the two-sided character* $\xi(T)$ *any system of numbers*

$$\delta(B) = \epsilon_\alpha r_\alpha \delta^{(\alpha)}(B) + \in_\beta r_\beta \delta^{(\beta)}(B) + \cdots + \epsilon_\kappa r_\kappa \delta^{(\kappa)}(B)$$

where the $\epsilon_\alpha, \epsilon_\beta, \ldots \epsilon_\kappa$ have the values ± 1.[42]

Hence there is an infinite number of complements assigned to each two-sided character $\xi(T)$. In any case, the numbers $\delta(B)$ satisfy the conditions (36)–(37); moreover, one obtains two adjunct (composed) characters $\theta(B)$ and $\overline{\theta}(B)$ of B_n with their sum being $\xi(B)$ by putting

$$\theta(B) = \tfrac{1}{2}[\xi(B) + \delta(B)], \qquad \overline{\theta}(B) = \tfrac{1}{2}[\xi(B) - \delta(B)]$$

If, in particular, $\xi(T) = \chi(T)$ is a simple character, $\theta(B)$ and $\overline{\theta}(B)$ become *actual* characters of B_n only if

$$\delta(B) = \pm[\psi(B) = \overline{\psi}(B)]$$

where $\psi(B)$ and $\overline{\psi}(B)$ have the same meaning as before. These two special complements of $\chi(T)$ are considered, as mentioned above, as not essentially distinct. Talking of *the* complement of a simple two-sided character, we mean one of those two complements.

Paragraph 17. If $\chi(T)$ is an arbitrary character of second kind of T_n, then with each permutation P of S_n

$$\chi(JP') = -\chi(P')$$

where P' is the element of T_n to be determined by the rules of Paragraph 11. Moreover, as, with each permutation being of the first kind, P' and JP' are conjugated elements of T_n and hence $\chi(JP') = \chi(P')$, it follows for any permutation of first kind that

$$\chi(JP') = \chi(P') = 0$$

It is therefore sufficient to name only the numbers $\chi(P')$ *for the permutations of second kind if* $\chi(T)$ *is a character of second kind.*

[42] If there is no two-sided character among those of (38), I say that *the complement of* $\xi(T)$ *is zero*.

As we have chosen the elements P' such that to two conjugated permutations of S_n correspond also two conjugated elements of T_n, the number $\chi(P')$ is only determined by the class of similar permutations of S_n which includes P.

Such a class is called even or odd depending on whether its permutations are even or odd. $[\alpha]$ denotes a class whose permutations exclusively consist of cycles of an odd order. If, among the cycles of a permutation p of $[\alpha]$, α_1 cycles are of the order 1, α_3 cycles of the order 3, etc., I put

$$[\alpha] = [\alpha_1, \alpha_3, \ldots] \qquad \text{and} \qquad \chi(P') = \chi_\alpha = \chi_{\alpha_1, \alpha_2, \ldots}$$

The class $[\alpha]$ contains

$$h_\alpha = \frac{n!}{1^{\alpha_1}\alpha_1!\ 3^{\alpha_3}\alpha_3!\ \ldots}$$

permutations and this also is the number of elements of T_n conjugated with P'. The number of classes $[\alpha]$ equals v_n. Noting that P and P^{-1} are similar permutations and that the order of P is odd, one sees that P' and P'^{-1} are conjugated elements of T_n. In our case, therefore, $\chi(P') = \chi(P'^{-1})$ and *all the numbers* χ_α *are real*.[43] Especially, if $\chi(T)$ is a *simple* character of second kind, (30) and (31) imply

$$\sum h_\alpha \chi_\alpha^2 = \frac{n!}{2^\epsilon} \tag{39}$$

where the sum goes over all v_n classes $[\alpha]$ and ϵ is 0 or 1 depending on whether $\chi(T)$ is a two-sided character or not. It also follows from (26) that

$$\sum h_\alpha \chi_\alpha \chi'_\alpha = 0 \tag{40}$$

where $\chi(T)$ and $\chi'(T)$ are two distinct characters not associated to each other.

Apart from the classes $[\alpha]$ which contain all the even permutations of second kind, we also have to consider those classes of S_n whose permutations can be decomposed into cycles of distinct orders. I designate such a class a (ν), (ρ), . . . and put

$$(\nu) = (\nu_1, \nu_2, \ldots \nu_m) \tag{41}$$

[43] Generally, $\chi(T)$ and $\chi(T^{-1})$ are conjugate complex numbers for any character. It is also easy to conclude that all the numbers χ_α are real; this, however, will be shown later in a different manner.

and

$$\chi(P') = \chi_{(\nu)} = \chi_{(\nu_1,\nu_2,\ldots\nu_m)}$$

if a permutation P of (ν) contains exactly m cycles of the order

$$\nu_1, \nu_2, \ldots \nu_m \qquad (\nu_1 > \nu_2 > \cdots > \nu_m > 1)$$

The number of classes (ν) also equals v_n, but those among them with which the $\nu_1, \nu_2, \ldots \nu_m$ are odd are also contained among the classes $[\alpha]$. The numbers $\chi_{(\nu)}$ have to be named only for the u_n *odd* classes (ν) because the remaining ones either appear among the numbers χ_α or are zero by themselves. If $\chi(T)$ is a two-sided character, $\chi_{(\nu)}$ also becomes zero with any odd class (ν). In this case, we will have to specify at least one complement $\delta(B)$ of $\chi(T)$. If P denotes the permutation

$$(1, 2, \ldots \nu_1)(\nu_1 + 1, \nu_1 + 2, \ldots \nu_1 + \nu_2) \cdots$$

of the class (41) and if P' is the fixed element of T_n as mentioned above, I put with (ν) being an *even* class

$$\delta(P') = \delta_{(\nu)} = \delta_{(\nu_1,\nu_2,\ldots\nu_m)}$$

Knowing the numbers $\delta_{(\nu)}$ *for all the* g_n *even classes* (ν), *one can specify all the other numbers* $\delta(B)$, *too, according to equations (34)–(37)*. In our case, we have to put $j = -1$.

Defining the number $n!/\nu_1\nu_2 \ldots \nu_m$ of permutations of the class (41) as $h_{(\nu)}$, one obtains with a *simple* character of second kind which is not two-sided the equation

$$\sum h_{(\nu)}\chi_{(\nu)}\overline{\chi}_{(\nu)} = \frac{n!}{2} \tag{42}$$

Similarly, with (35), the complement of a two-sided character becomes

$$\sum h_{(\nu)}\delta_{(\nu)}\overline{\delta}_{(\nu)} = n! \tag{43}$$

In (42), the sum contains all the odd classes (ν) and in (43) all the even ones. Moreover, $\overline{\chi}_{(\nu)}$ and $\overline{\delta}_{(\nu)}$ are the numbers complex conjugated to $\chi_{(\nu)}$ and $\delta_{(\nu)}$.

I derive two other formulas which will be important in the following.

Generally, with each permutation P of S_n,

$$\sum_{\chi} \chi(P')\chi(P'^{-1}) = \frac{2n!}{h_P}$$

where the sum includes all the simple characters of first and second kind of

T_n and h_P is the number of elements of T_n conjugated to P'.[44] If P is a permutation of second kind, then h_P also denotes the number of permutations of S_n similar to P. As the $n!$ numbers $\chi(P') = \chi(P)$, with each character of first kind, generate a character of S_n it follows that

$$\sum_{\chi}{}' \chi(P')\chi(P'^{-1}) = \frac{n!}{h_P}$$

where the sum includes all the characters of the first kind. Hence,

$$\sum_{\chi}{}'' \chi(P')\chi(P'^{-1}) = \frac{n!}{h_P}$$

where χ becomes any of the $v_n + u_n$ characters of the second kind. I will call the v_n simple characters of the second kind among which are no two associated to each other

$$\chi^{(1)}(T),\ \chi^{(2)}(T),\ \ldots\ \chi^{(v_n)}(T)$$

Furthermore, let ϵ_ρ be equal to 0 or 1, depending on whether $\chi^{(\rho)}(T)$ is a two-sided character or not. The last equation can then be rewritten, if P is contained in the class $[\alpha]$, as

$$\sum_{\rho} 2^{\epsilon_\rho}\chi_\alpha^{(\rho)2} = \frac{n!}{h_\alpha} \tag{44}$$

However, if $[\alpha]$ and $[\beta]$ are different classes, one obtains similarly

$$\sum_{\rho} 2^{\epsilon_\rho}\chi_\alpha^{(\rho)}\chi_\beta^{(\rho)} = 0 \tag{45}$$

5. ON THE COLLINEATION GROUPS BELONGING TO THE CHARACTERS OF THE GROUPS T_n AND B_n

Paragraph 18. As in Paragraph 13, let H denote one of the groups T_n or B_n and G be either S_n or A_n. If g is the order of G, the order h of H equals $2g$.

Again, consider a simple character $\chi(R)$ of H and a representation Δ of H belonging to $\chi(R)$ by matrices (R). For any permutation P of G, denote the collineation determined by the matrices (P') and (JP') by P and the group generated by this collineation by K.

It has to be mentioned first that if $n \geq 4$ and $\chi(R)$ is a character of *second* kind, the g collineations P must be distinct. If this were not the case, there would at least be one permutation P distinct from E such that $P = E$

[44] This is one of the basic equations of the theory of group characters. Compare to my work, Neue Begruendung der Theorie der Gruppencharaktere, Equation (XIV).

and these permutations would build an invariant subgroup F of G. If $n > 4$, it would follow that $F = G$ or $G = S_n$, as A_n is a simple group and S_n contains only this one invariant subgroup A_n. If $n = 4$, the group of the four elements

$$E, A = (1, 2)(3, 4), \qquad B = (1, 3)(2, 4), \qquad C = (1, 4)(2, 3)$$

would have to be considered for F. In any case, F contains the permutations A and B. For the corresponding elements A' and B' of H, the matrices (A') and (B') in our representation Δ would only differ by a constant factor and, hence, commute. However, if $T_1, T_2, \ldots$ denote the elements generating the group T_n,

$$A' = J^{\alpha} T_1 T_3, \qquad B' = J^{\beta} T_2 T_1 T_3 T_2$$

and this yields $A'B' = JB'A'$. According to our assumption about the character $\chi(R)$, in any case $(J) = -(E)$, it follows that $(A')(B') = -(B')(A')$, which leads to a contradiction.

Therefore, the collineation group K belonging to a character of the second kind of H is always isomorphic to the group G if $n \geq 4$.

Similarly, it can be concluded that the group K is not isomorphic in the first degree to the group S if the order f of a simple character of first kind of H equals 1 or $G = S_4$ and $f = 2$.

Paragraph 19. Let $\overline{\chi}(R)$ denote a simple character of H different from $\chi(R)$ and corresponding to the representation $\overline{\Delta}$ of H by the matrices $(\overline{R})$. The corresponding collineation group shall be called $\overline{K}$; furthermore, $\{\overline{P}\}$ shall be the substitution of $\overline{K}$ corresponding to the permutation P of S.

We will examine the conditions under which the groups K and $\overline{K}$ equal each other, apart from the ordering.

We have to distinguish between two cases.

(a) Let $\{\overline{P}\} = \{P\}$ for each permutation P of G. Then, the coefficient matrices of these two collineations differ only by a number and therefore, with any element R of H,

$$(\overline{R}) = \zeta_R \cdot (R)$$

which yields

$$\overline{\chi}(R) = \zeta_R \cdot \chi(R) \tag{46}$$

where the ζ_R are certain numbers. The first equation implies with any two elements R and S of H

$$\zeta_R \zeta_S = \zeta_{RS}$$

i.e., the numbers ζ_R build a linear character of H.[45] If $H = T_n$, the commutator of H is the subgroup B_n with index 2. Apart from the main character $\zeta_R = 1$, which is of no interest, there is only one other linear character which can be obtained by putting $\zeta_R = 1$ or $\zeta_R = -1$, depending on whether R is contained in B_n or not. Equation (46) then shows us that χ *and* $\overline{\chi}$ *become associated characters*. Also, one concludes immediately that *the collineation groups belonging to two associated characters of* T_n *are to be considered as not distinct*.

Let H now be the group B_n. For $n > 4$, the commutator of B_n contains all the elements of the goup and it follows that B_n has only the linear character $\zeta_R = 1$, which will be excluded again. However, if $n = 4$, B_n possesses three linear characters $\zeta_0(R)$, $\zeta_1(R)$, $\zeta_2(R)$, which are determined by

$$\zeta_\alpha(T_2T_1) = \rho^\alpha, \qquad \zeta_\alpha(T_3T_1) = 1$$

where ρ is a primitive cubic root of the identity. The group B_4 is an exception which has to be considered in the following.

(b) In this case, let the substitution $\{\overline{P}\}$ of $\overline{K}$ be equal to the substitution $\{P_1\}$ of K, where P_1 means a permutation of G which not necessarily equals P. We obviously obtain an automorphism A of G by assigning the permutation P_1 to P. First, if A is an inner automorphism of G, there exists a permutation H within G such that $H^{-1}PH = P_1$. This, however, leads to case (a) if one substitutes K by the group equivalent to it which is generated by the linear transformation H.

Hence, A is an outer automorphism of G. If $G = S_n$, we only have to consider the case where $n = 6$, as S_n is a complete group with $n = 6$. Hence, $G = A_n$. If $n = 6$, again A can only be an automorphism obtained by transforming all the permutations of A_n by an odd permutation U. This yields $P_1 = U^{-1}PU$; according to the assumption, the collineations $\{U^{-1}PU\}$ and $\{\overline{P}\}$ are the same. Designating the element U' belonging to U in T_n by C, one discovers that the representations Δ and $\overline{\Delta}$ of the group $H = B_n$ are connected such that with any element R of B_n,

$$(C^{-1}RC) = \zeta_R \cdot (\overline{R}) \tag{47}$$

where ζ_R is a constant. These numbers ζ_R build another linear character of B_n. Neglecting the case where $n = 4$, it follows that $\zeta_R = 1$. Therefore, equation (47) implies

[45] Compare to Frobenius, Ueber die Primfaktoren der Gruppendeterminante, *Sitzungber. K. Preuss. Akad. Berlin* (1896), p. 1343.

$$\chi(C^{-1}RC) = \bar{\chi}(R)$$

i.e., χ and $\bar{\chi}$ are adjunct characters of B_n (compare to Paragraph 15). Conversely, with χ and $\bar{\chi}$ being adjunct characters of B_n, the group K becomes equal to $\bar{K}$ or to a group equivalent to $\bar{K}$ if one permutes the elements of K with the automorphism A of A_n. In this case I call K and $\bar{K}$ *adjunct groups* (see Introduction).

In the previously excluded case $n = 6$, either with S_6 or A_6, one has to consider the well-known automorphism A which assigns a permutation of the form $(\alpha\beta\gamma)(\delta\epsilon\eta)$ to each cycle of the order three. Moreover, as we will see later, in each of the groups T_6 and B_6, there exist certain pairs of characters χ and $\bar{\chi}$ whose collineation groups are transformed into each other by the automorphism A.

Paragraph 20. If one wants to know only those irreducible collineation groups which are isomorphic to the groups S_n or A_n and cannot be written as groups of $n!$ and $n!/2$, respectively, homogeneous linear substitutions, one has to consider only the simple characters of second kind of T_n or B_n. Furthermore, two associated characters within the group T_n and two adjunct ones within B_n are not essentially distinct. With the results on the number of characters of the second kind within the groups T_n and B_n obtained above, we can state the theorem announced in the Introduction:

VI. For $n > 3$ and not 6, the number of essentially different irreducible collineation groups isomorphic to S_n which cannot be written as groups of $n!$ homogeneous linear substitutions equals the number v_n of decompositions of n into distinct summands. If $n > 4$ and not 6 or 7, the corresponding number with the group A_n also equals v_n.

The group S_6 is an exception as a matter of the outer automorphism mentioned above. Here, as I emphasized in the Introduction, only three of the $v_n = 4$ collineation groups are essentially different. For the group A_4, the $v_4 = 2$ collineation groups reduce to only one group because of the appearing of linear characters within the group B_4. The cases $n = 6$ and $n = 7$ play an important role only with the group A_n as the groups B_6 and B_7 are no longer the representation groups of A_6 and A_7.

6. THE PRINCIPAL REPRESENTATION OF SECOND KIND OF THE GROUP T_n

Paragraph 21. In this paragraph, I will set up and examine the collineation group of order $2^{[n-1/2}$ isomorphic to S_n which I mentioned in the Introduction.

For

$$A = (a_{\chi\lambda}), \qquad B = (b_{\mu\nu})$$

two matrices of ranks p and q, the matrix of rank pq

$$\begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1p}B \\ a_{21}B & a_{22}B & \cdots & a_{2p}B \\ \cdots & \cdots & \cdots & \cdots \\ a_{p1}B & a_{p2}B & \cdots & a_{pp}B \end{pmatrix}$$

will be called $A \times B$. If C denotes a third matrix of rank r, then

$$(A \times B) \times C = A \times (B \times C)$$

This matrix of rank pqr is called $A \times B \times C$. Analogously, with m matrices $A_1, A_2, \ldots A_m$ of arbitrary ranks, we define the symbol

$$A_1 \times A_2 \times \cdots \times A_m \tag{48}$$

If the $A_1, A_2, \ldots A_m$ are all equal to a matrix A, I will designate (48) as $\Pi_m A$. The trace of the matrix (48) equals the product of the traces of the matrices $A_1, A_2, \ldots A_m$. Moreover, with m arbitrary matrices $B_1, B_2, \ldots B_m$ and B_μ being of the same rank as A_μ, we obtain[46]

$$(A_1 \times A_2 \times \cdots \times A_m)(B_1 \times B_2 \times \cdots \times B_m) = A_1B_1 \times A_2B_2 \times \cdots \times A_1B_1 \tag{49}$$

Consider next the four matrices with rank two:

$$F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These matrices satisfy the following conditions:

$$A^2 = F, \qquad B^2 = -F, \qquad C^2 = F$$
$$AB = -BA = -C, \qquad BC = -CB = -A, \qquad CA = -AC = B \tag{50}$$
$$CBA = F$$

Next, form, for an arbitrary m, the matrices of rank 2^m,

$$M_1 = \Pi_m A, \qquad M_2 = \Pi_{m-1} A \times B, \qquad M_3 = \Pi_{m-1} A \times C, \ldots$$
$$M_{2\nu} = \Pi_{m-\nu} A \times B \times \Pi_{\nu-1} F, \qquad M_{2\nu+1} = \Pi_{m-\nu} A \times C \times \Pi_{\nu-1} F, \ldots$$

[46] Compare to A. Hurwitz, Zur Invariantentheorie, *Math. Ann.* **45**, 381, and my dissertation, Paragraph 6.

$$M_{2m} = B \times \Pi_{m-1} F, \qquad M_{2m+1} = C \times \Pi_{m-1} F$$

These $2m + 1$ matrices satisfy the equations, according to (49) and (50),

$$M_{2\nu}^2 = -E, \qquad M_{2\nu+1}^2 = E \tag{51}$$

$$M_\chi M_\lambda = -M_\lambda M_\chi \tag{52}$$

$$M_{2m+1} M_{2m} \cdots M_2 M_1 = E, \tag{53}$$

where $E = \Pi_m F$ denotes the identity matrix of rank 2^m.

Equations (51) and (52) yield that any product of the $2m$ matrices M_1, $M_2, \ldots M_{2m}$ equals one of the

$$1 + \binom{2m}{1} + \binom{2m}{2} + \cdots \binom{2m}{2m} = 2^{2m}$$

matrices

$$E,\ M_1,\ M_2,\ \ldots\ M_{2m},\ M_1\ M_2,\ M_1\ M_3,\ \ldots\ M_{2m-1}\ M_{2m},\ \ldots\ M_1\ M_2\ \ldots\ M_{2m} \tag{54}$$

neglecting the sign. Without great effort, one can also see that these matrices, apart from the sign, are in accordance with the 4^m matrices which are obtained by substituting for the $A_1, A_2, \ldots A_m$ in (48) the matrices F, A, B, and C in any possible manner. As the traces of F, A, B, and C are all zero, it follows that among the matrices (54)—I will call them $X_0, X_1, X_2, \ldots$—only the first one has a nonzero trace; the trace of $X_0 = E$, however, equals 2^m. Moreover, the matrices $X_0, X_1, X_2, \ldots$ are reproduced, apart from the sign, by multiplication, namely,

$$X_\chi^2 = \pm E, \qquad X_\chi X_\lambda = \pm X_\mu$$

where μ is not zero.

This implies that the $X_0, X_1, X_2, \ldots$ are linearly independent, i.e., if

$$a_0 X_0 + a_1 X_1 + a_2 X_2 + \ldots = 0$$

multiplication with X_χ yields

$$a_0 X_0 X_\chi + \cdots + a_{\chi-1} X_{\chi-1} X_\chi \pm a_\chi E + a_{\chi+1} X_{\chi+1} X_\chi + \cdots = 0$$

As the trace of the matrix on the left-hand side equals $\pm 2^m a_\chi$, it follows that $a_\chi = 0$. Considering also that the number of matrices $X_0, X_1, X_2, \ldots$ equals the square of their rank, one discovers that *any* matrix of the rank 2^m can be represented as a linear homogeneous combination of the $X_0, X_1, X_2, \ldots$. *Hence, the matrices* $M_1, M_2, \ldots M_{2m}$ *generate an irreducible group*.[47]

Paragraph 22. With the help of the matrices $M_1, M_2, \ldots M_{2m+1}$, we can now make up a representation of second kind of the group T_n.

Under the assumption that $m = [n - 1/2]$, i.e., $n = 2m + 1$ or $n = 2m + 2$, I put

$$T_\lambda = a_{\lambda-1} M_{\lambda-1} + b_\lambda M_\lambda \qquad (\lambda = 1, 2, \ldots n - 1) \tag{55}$$

where

$$a_{2\nu} = -\frac{\sqrt{\nu}}{\sqrt{2\nu + 1}}, \qquad b_{2\nu+1} = \frac{i\sqrt{\nu + 1}}{\sqrt{2\nu + 1}}$$

$$a_{2\nu+1} = -\frac{i\sqrt{2\nu + 1}}{2\sqrt{2\nu + 1}}, \qquad b_{2\nu+2} = \frac{\sqrt{2\nu + 3}}{2\sqrt{\nu + 1}} \qquad (\nu = 0, 1, 2, \ldots)$$

Here all the roots are to be taken positive. Therefore,

$$T_1 = iM_1, \qquad T_2 = -\frac{i}{2} M_1 + \frac{\sqrt{3}}{2} M_2, \qquad T_3 = -\frac{1}{\sqrt{3}} M_2 + \frac{i\sqrt{2}}{\sqrt{3}} M, \ldots .$$

The values a_λ and b_λ satisfy the equations

$$b_\lambda^2 - a_{\lambda-1}^2 = (-1)^\lambda, \qquad a_\lambda b_\lambda = \frac{(-1)^{\lambda-1}}{2}$$

Using these equations and the formulas (51) and (52), the relations

$$T_\alpha^2 = -E, \qquad T\beta T_{\beta+1} + T\beta + 1T_\beta = E, \tag{56}$$

$$T_\gamma T_\delta = -T_\delta T_\gamma, \qquad \begin{bmatrix} \alpha = 1, 2, \ldots n - 1 \\ \beta = 1, 2, \ldots n - 2 \\ \gamma = 1, 2, \ldots n - 3 \\ \delta \geq \gamma + 2 \end{bmatrix}$$

follow easily and this implies

$$(T_\beta T_{\beta+1})^3 = -E$$

Putting $J = -E$, one sees that the matrices $J, T_1, T_2, \ldots T_{n-1}$ satisfy the relations (II) which define the group T_n. Therefore they generate a representation of T_n by matrices of rank $2^{[n-1/2]}$, and as $J = -E$, this is a representation of second the kind. I denote this representation as Δ_n and call it the *main representation of the second kind of the group* T_n.

[47] This group is a finite group of the order 2^{2m+1}.

The fact that the representation Δ_n is irreducible can be seen in this way: As $T_1 = iM_1$, any one of the matrices $M_1, M_2, \ldots M_{2m}$ can be represented as a linear homogeneous combination of $T_1, T_2, \ldots T_{n-1}$. If the group generated by the T_α was reducible, the group generated by the M_α would also be reducible. However, as we have seen, this is not the case.

Equations (56) give rise to the following consideration. Only using these equations, one can represent any product $T_\alpha T\beta T_\gamma \cdots$ as a linear homogeneous combination of the 2^{n-1} special products

$$E,\ T_1, \ldots T_{n-1},\ T_1T_2,\ T_1T_3, \ldots T_{n-2}T_{n-1},\ T_1T_2T_3, \ldots T_1T_2 \ldots T_{n-1} \tag{57}$$

where the coefficients can only be integers. Moreover, there cannot be derived a linear homogeneous relation with constant coefficients between the products (57) from equations (56). Actually, these equations can be satisfied by the matrices (55). Among the linear combinations of the products of these matrices are, as we have already seen, the M_α and therefore also the 2^{2m} linearly independent matrices (54). For odd n, $2^{2m} = 2^{n-1}$ and the 2^{n-1} products (57) cannot be linearly dependent in this case. For even n, however, add another T_n to the $T_1, T_2, \ldots T_{n-1}$ and add to equations (56) the equations

$$T_n^2 = -E, \qquad T_{n-1}T_n + T_nT_{n-1} = E, \tag{58}$$

$$T_\beta T_n = -T_nT_\beta \qquad (\beta = 1, 2, \ldots n-2)$$

As $n + 1$ is odd, equations (56) and (58) do not imply a linear homogeneous relation between the 2^n products

$$E,\ T_1, \ldots T_{n-1},\ T_1T_2,\ T_1T_3, \ldots T_{n-2}T_{n-1},\ T_1T_2T_3, \ldots T_1T_2, \ldots T_{n-1}$$

and therefore no relation between the products (57) can be derived.

Denoting the products (57) as

$$A_1, A_2, \ldots A_{2^{n-1}}$$

we find that (56) yields equations of the kind

$$A_\chi T_\alpha = \sum_{\lambda=1}^{2^{n-1}} t_{\chi\lambda}^{(\alpha)} A_\lambda$$

where the $t_{\chi\lambda}^{(\alpha)}$ denote certain integers. The matrices

$$\overline{T}_\alpha = (t_{\chi\lambda}^{(\alpha)})$$

of rank 2^{n-1} obviously generate a group isomorphic (in the first degree) with the group T_n:

The group T_n can be represented as a linear homogeneous group of rank 2^{n-1} with integer coefficients.

Paragraph 23. In the following, I will calculate the (simple) character $\chi(T)$ belonging to the representation Δ_n, which I call the *main character of second kind*.

I start with this statement: Let X be a matrix of the form $xE + \Sigma\, x_\alpha P_\alpha$, where the P_α are products of k certain matrices of the set $M_1, M_2, \ldots, M_{2m}$. Similarly, let $Y = yE + \Sigma\, y_\beta Q_\beta$, where the Q_β denote products of all other matrices of this set. Here none of the products P_α and Q_β may equal $\pm E$. Then all the products $P_\alpha Q_\beta$ are distinct from $\pm E$. Therefore, according to Paragraph 21, the traces of all the matrices P_α, Q_β, $P_\alpha Q_\beta$ equal zero. As the trace of the identity matrix E of rank 2^m equals 2^m, it follows that *the traces of the matrices X, Y, XY take the values* $2^m x$, $2^m y$, $2^m xy$.

Next, let P be a permutation (of second kind), consisting of σ cycles, of the form

$$(1, 2, \ldots \lambda_1)(\lambda_1 + 1, \lambda_1 + 2, \ldots \lambda_1 + \lambda_2) \ldots \tag{59}$$

where we assume that $\lambda_1 \geq \lambda_2 \geq \cdots$. To this permutation of S_n there corresponds in T_n the element

$$P' = C_{\lambda_1} C_{\lambda_2} \ldots$$

where

$$C_{\lambda_1} = T_{\lambda_1 - 1} T_{\lambda_2 - 2} \cdots T_1, \qquad C_{\lambda_2} = T_{\lambda_1+\lambda_2-1} T_{\lambda_1+\lambda_2-2} \cdots T_{\lambda_1+1}, \ldots$$

and, if $\lambda_\alpha = 1$, C_{λ_α} denotes the identity E of T_n. The matrices of Δ_n corresponding to the elements P' and C_{λ_α} of T_n shall be designated with the same letters. For odd n, i.e., $n = 2m + 1$, M_{2m+1} appears in none of the matrices in (55) and can be neglected if the C_{λ_α} are expressed in terms of the M_χ. However, if $n = 2m + 2$, M_{2m+1} appears only in T_{n-1} and has to be considered only if P denotes the cycle $(1, 2, \ldots n)$, i.e.,

$$P' = T_{n-1} T_{n-2} \cdots T_1$$

Moreover, C_{λ_1} can be expressed by the products of the matrices

$$M_1, M_2, \ldots M_{\lambda_2 - 1}$$

and similarly, C_{λ_2} by the products of the matrices

$$M_{\lambda_1}, M_{\lambda_1+1}, \ldots M_{\lambda_1+\lambda_2-1}$$

and so on. In light of the statement made before, one can see that *if the trace of C_{λ_α} equals $2^m c_\alpha$, the trace of P' takes the value $2^m c_1 c_2 \cdots$.*

We therefore only have to calculate the trace of one element of the kind

$$C = T_\beta T_{\beta-1} \cdots T_{\beta-\alpha}$$

It follows that

$$C = (a_{\beta-1}M_{\beta-1} + b_\beta M_\beta)(a_{\beta-2}M_{\beta-2} + b_{\beta-1}M_{\beta-1}) \times \cdots (a_{\beta-\alpha-1}M_{\beta-\alpha-1} + b_{\beta-\alpha}M_{\beta-\alpha})$$

Since we are only interested in the trace of C, doing the multiplication, one has to consider solely those factors which have the form cE. For an even number $\alpha + 1$ of factors T_λ of C, the required form has only this one part

$$a_{\beta-1}b_{\beta-1}M^2_{\beta-1} \cdot a_{\beta-3}b_{\beta-3}M^2_{\beta-3} \cdots a_{\beta-\alpha}b_{\beta-\alpha}M^2_{\beta-\alpha}$$

As

$$M^2_\lambda = (-1)^{\lambda-1}E, \qquad a_\lambda b_\lambda = \frac{(-1)^{\lambda-1}}{2}$$

the trace of C equals $2^{m-(\alpha+1)/2}$. However, if $\alpha + 1$ is odd, the trace of C equals zero with the only exception if $n = 2m + 2$ and

$$\begin{aligned} C &= T_{n-1}T_{n-1} \cdots T_1 \\ &= (a_{2m}M_{2m} + b_{2m+1}M_{2m+1})(a_{2m-1}M_{2m-1} + b_{2m}M_{2m}) \\ &\quad \times \cdots (a_1M_1 + b_2M_2) \cdot b_1M_1 \end{aligned}$$

In this case, the expansion of C contains the factor

$$b_{2m+1}M_{2m+1} \cdot b_{2m}M_{2m} \cdots b_2M_2 \cdot b_1M_1$$

which, according to (53), equals

$$b_1b_2 \cdots b_{2m+1} \cdot E$$

This factor has the value

$$i \cdot \frac{\sqrt{3}}{2} \cdot \frac{i\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{5}}{2\sqrt{2}} \cdots \frac{\sqrt{2m+1}}{2\sqrt{m}} \cdot \frac{i\sqrt{m+1}}{\sqrt{2m+1}} = \frac{i^{m+1}\sqrt{m+1}}{2^m}$$

Hence, the trace of C equals

$$i^{m+1}\sqrt{m+1} = \sqrt{(-1)^{n/2}n/2}$$

If all the orders $\lambda_1, \lambda_2, \ldots \lambda_\rho$ of the cycles of the permutation P examined earlier are odd, the trace $2^m c_\alpha$ of C_{λ_α} becomes $2^{m-(\lambda_\alpha-1)/2}$, and therefore the trace $\chi(P')$ of the matrix P equals

$$2^{m-(\lambda_1-1)/2-\lambda_2-12-\cdots-\lambda_\rho-12} = 2^{(2m-n+\rho)/2} = 2^{[\rho-1/2]}$$

On the other hand, if only one of the λ_α is even, $\chi(P')$ becomes zero except that n is even and $P = (1, 2, \ldots n)$. In this case,

$$\chi(P') = \sqrt{(-1)^{n/2} n/2}$$

Using the notation introduced in Paragraph 17, one obtains:

VII. For $[\alpha]$ a class of similar permutations of S_n which can be decomposed into σ_α cycles of odd order, we have for the main character of the second kind of T_n, $\chi(T)$,

$$\chi_\alpha = 2^{[\sigma_\alpha - 1/2]}$$

If n is odd, $\chi(T)$ is a two-sided character. However, if n is even, $\chi(T)$ is not a two-sided character and we obtain for the class (n) of cycles of nth order

$$\chi_{(n)} = \sqrt{(-1)^{n/2} n/2}$$

For any other class (ν), however,

$$\chi_{(\nu)} = 0$$

Paragraph 24. If n is odd, we also have to determine the simple characters $\psi(B)$ and $\overline{\psi}(B)$ of B_n belonging to $\chi(T)$, as $\chi(T)$ then is a two-sided character. According to the above, it is sufficient to determine the complement $\delta(B) = \psi(B) - \overline{\psi}(B)$ of $\chi(T)$.

In order to solve this problem, one has to keep in mind that, if $n = 2m + 1$, the the matrix M_{2m+1} does not contain the elements $T_1, T_2, \ldots T_{n-1}$ of our representation Δ_n. As

$$M_{2m+1} M_\lambda = -M_\lambda M_{2m+1}$$

it follows that

$$M_{2m+1}^{-1} T_\lambda M_{2m+1} = -T_\lambda$$

Furthermore, $M_{2m+1}^2 = E$ and this implies that M_{2m+1} plays the same role in our representation Δ_n as the matrix H in the two-sided representation of Paragraph 16. In order to determine $\delta(B)$, one therefore has to calculate the trace of the matrix $M_{2m+1}\, P'$ only for the even P. Again, we can restrict ourselves to permutations P of the form (59).

As, according to equations (51)–(53),

$$M_{2m+1} = \pm M_1 M_2 \cdots M_{2m}$$

the trace of a product

$$M_{2m+1}M_{\alpha}M_{\beta}\cdots$$

is always zero, with α, β, . . . being any indices of the series 1, 2, . . . $2m$ whose number is smaller than $2m$. This implies immediately that the trace of $M_{2m+1}P'$ equals zero if P is not $(1, 2, \ldots n)$. Again, in this case, however,

$$\begin{aligned} P' &= T_{n-1}T_{n-2}\cdots T_1 \\ &= (a_{2m-1}M_{2m-1} + b_{2m}M_{2m})(a_{2m-2}M_{2m-2} + b_{2m-1}M_{2m-1}) \\ &\quad \times \cdots (a_1M_1 + b_2M_2)\cdot b_1M_1 \end{aligned}$$

and the trace $\delta(P')$ of $M_{2m+1}P'$ equals the trace of

$$M_{2m+1}\cdot b_{2m}M_{2m}\cdot b_{2m-1}M_{2m-1}\cdots b_2M_2\cdot b_1M_1 = b_1b_2\cdots b_{2m}E$$

i.e.,

$$2^m b_1b_2\cdots b_{2m} = i^m\sqrt{2m+1} = \sqrt{(-1)^{(n-1)/2}n}$$